ON FUZZY ECONOMIC ORDER QUANTITY USING POSSIBILISTIC APPROACH

Recently, fuzzy sets theory has been used to generalize, under fuzzy environment, the economic order quantity (EOQ) inventory model. To widen the application of this approach, this paper considers the parameters in the EOQ model as trapezoidal fuzzy numbers (Tr.F.N.’s) and derives certain results using possibilistic approach. Finally certain comments are suggested for further studies in supply chain management.

Introduction

The well-known EOQ inventory model has been extended to fuzzy versions by Park (1987), Chen and Wang (1996), Roy and Maiti (1997), Yao et al. (2000), Yao et al. (2003) and Chang (2004) among others. In this paper, we develop a fuzzy possibilistic version of the EOQ model under the assumptions that the parameters involved are trapezoidal fuzzy numbers.
Figure 1

Inventory Model with Constant Demand and Instantaneous Supply

Source: Sharma (1997)
This paper is organized as follows. We summarize the preliminaries in Section 1. In Section 2, we discuss the fuzzy possibilistic EOQ inventory model along with some important results. The approach used to derive the optimal economic order quantity is somewhat close to the method adopted by Hsieh (2002) for deriving the optimal order quantity. It is of interest to note here that Hsieh (2002) used an approximation to defuzzify the fuzzy total cost, whereas, our approach does not use any kind of approximation. Concluding remarks are made in Section 4.

Preliminaries and Notation

Before discussing the fuzzy possibilistic EOQ inventory model, we introduce some definitions and with relevant operations. Most of these related definitions and properties may be found in Bector and Chandra (2005), Kaufmann and Gupta (1985), Zimmermann (2001).

Let \( \mathbb{R} \) be the set of real numbers, \( \mathbb{R}^+ \) be the set of positive real numbers, and \( X \in \mathbb{R} \).

**Definition 1.1**
Fuzzy set $A$ in $X \subseteq \mathbb{R}$, the set of real numbers, is a set of ordered pairs $A = \{(x, \mu(x) : x \in X\}$, where $\mu(x)$ is the membership function or grade of membership, or degree of compatibility or degree of truth of $x \in X$ which maps $x \in X$ on the real interval $[0, 1]$.

**Definition 1.2**

If $\text{Sup } \mu(x) = 1$, $x \in \mathbb{R}$, then the fuzzy set $A$ is called a normal fuzzy set in $\mathbb{R}$.

**Definition 1.3**

The crisp set of elements that belong to the fuzzy set $A$ at least to the degree $\alpha$ is called the $\alpha$-level set (or $\alpha$-cut), i.e. $A(\alpha) = \{x \in X | \mu(x) \geq \alpha, \alpha \in \mathbb{R}^+\}$. If the set $A'(\alpha) = \{x \in X | \mu(x) > \alpha, \alpha \in \mathbb{R}^+\}$, then $A'(\alpha)$ is called strong $\alpha$-level set (or strong $\alpha$-cut).

**Definition 1.4**

A fuzzy set $A$ is said to be a convex set if $\mu(\lambda x_1 + (1 - \lambda)x_2) \geq \min (\mu(x_1), \mu(x_2))$, $x_1, x_2 \in X$ and $\lambda \in [0, 1]$. Alternatively, a fuzzy set $A$ is convex if its every $\alpha$-level sets is a convex set.

**Definition 1.5**

A fuzzy set $A$, which is both convex and normal, is defined to be a fuzzy number on the universal set $\mathbb{R}$.

**Definition 1.6**

A Trapezoidal Fuzzy Number (Tr.F.N.) is represented completely by a quadruplet $A = (a_1, a_2, a_3, a_4)$, where $a_1 < a_2 < a_3 < a_4 \in X$ with membership function $\mu(x)$ given by

$$
\mu(x) = \begin{cases} 
0 & x \leq a_1 \\
\frac{x - a_1}{a_2 - a_1} & a_1 \leq x \leq a_2 \\
1 & a_2 \leq x \leq a_3 \\
\frac{x - a_3}{a_4 - a_3} & a_3 \leq x \leq a_4 \\
0 & x \geq a_4 
\end{cases}
$$

(1.1)

Alternatively Kaufmann and Gupta (1985), p.26,27, defining the interval of confidence at level $\alpha$ as $A(\alpha) = [p_1(\alpha), p_2(\alpha)]$, and setting $\frac{x - a_1}{a_2 - a_1} = \alpha$, and $\frac{x - a_3}{a_4 - a_3} = \alpha$, we get $p_1(\alpha) = a_1 + \alpha(a_2 - a_1)$ and $p_2(\alpha) = a_4 + \alpha(a_3 - a_4)$. Thus, we get the $\alpha$-cut representation of the Tr.F.N. $[a_1, a_2, a_3, a_4]$, where $a_1 < a_2 < a_3 < a_4 \in X$, as

$$
A(\alpha) = [p_1(\alpha), p_2(\alpha)] = [a_1 + \alpha(a_2 - a_1), a_4 + \alpha(a_3 - a_4)], \forall \alpha \in (0, 1).
$$

(1.2)
Possibilistic Mean, Variance and Covariance of Fuzzy Numbers

As in Appadoo et al. (2008) and Carlsson and Fuller (2001) we use the following equalities given in (1.3) and (1.4) in deriving some of the results in the lot size inventory model.

\[
\text{Possibility}[A \leq p_1(\alpha)] = \pi([-\infty, p_1(\alpha)]) = \sup_{u \leq p_1(\alpha)} A(u) = \alpha \\
\text{Possibility}[A \geq p_2(\alpha)] = \pi[p_2(\alpha), \infty) = \sup_{u \geq p_2(\alpha)} A(u) = \alpha
\]\n
(1.3) \hspace{1cm} (1.4)

For fuzzy number \(A(\alpha) = [p_1(\alpha), p_2(\alpha)], \alpha \in [0, 1]\) and \(B(\alpha) = [q_1(\alpha), q_2(\alpha)], \alpha \in [0, 1]\) Carlsson and Fuller (2001) define crisp lower possibilistic mean value \(E_L(A)\), crisp upper possibilistic mean value \(E_R(A)\) and crisp possibilistic mean value \(E(A)\). Consider a fuzzy number \(A\) whose \(\alpha\)-cut is written as \(A(\alpha) = [p_1(\alpha), p_2(\alpha)], \alpha \in \{0, 1\}\). Carlsson and Fuller (2001), define the lower and upper possibilistic mean \(E_L(A)\) and \(E_R(A)\), respectively, and the possibilistic mean \(E(A)\) of a fuzzy number \(A\) as follows.

\[
E_L(A) = \frac{\int_0^1 \text{Poss}[A \leq p_1(\alpha)] \min[A(\alpha)] \, d\alpha}{\int_0^1 \text{Poss}[A \leq p_1(\alpha)] \, d\alpha} = \frac{\int_0^1 \alpha \, p_1(\alpha) \, d\alpha}{\int_0^1 \alpha \, d\alpha}
\]

(1.5)

and

\[
E_R(A) = \frac{\int_0^1 \text{Poss}[A \geq p_2(\alpha)] \max[A(\alpha)] \, d\alpha}{\int_0^1 \text{Poss}[A \geq p_2(\alpha)] \, d\alpha} = \frac{\int_0^1 \alpha \, p_2(\alpha) \, d\alpha}{\int_0^1 \alpha \, d\alpha}
\]

(1.6)

\[
E(A) = \frac{1}{\int_0^1 \alpha \, d\alpha} \int_0^1 \alpha (p_1(\alpha) + p_2(\alpha)) \, d\alpha = \frac{E_L(A) + E_R(A)}{2}
\]

(1.7)

In view of expression (1.5) and (1.6), the interval valued possibilistic mean values of fuzzy number \(A\), Carlsson and Fuller (2003) can be written as

\[
IVPM(A) = [E_L(A), E_R(A)]
\]

(1.8)

The possibilistic variance of \(A\) is given by

\[
Var(A) = \int_0^1 \frac{1}{2} \alpha (p_2(\alpha) - p_1(\alpha))^2 \, d\alpha
\]

The possibilistic covariance between two fuzzy numbers, \(A\) and \(B\), is

\[
Cov(A, B) = \frac{1}{2} \int_0^1 \alpha \, [(p_2(\alpha) - p_1(\alpha))(q_2(\alpha) - q_1(\alpha))] \, d\alpha.
\]
Deterministic Economic Order Quantity (EOQ) Model

We now consider the classical inventory model in which we assume the parameters as crisp. The only variable in the classical EOQ model is \( Q; C_c, C_o, \) and \( D \) are constant parameters. In other words, demand is known with certainty. Demand is constant and replenishment is assumed to be instantaneous. Thus, Harris (1990)

\[
\text{Annual Carrying Cost} = C_c \frac{Q}{2} \quad (2.1)
\]
\[
\text{Annual Ordering Cost} = C_o \frac{D}{Q} \quad (2.2)
\]

Therefore,

\[
\text{Total Cost} = \text{Annual Carrying Cost} + \text{Annual Ordering Cost} = C_c \frac{Q}{2} + C_o \frac{D}{Q} \quad (2.3)
\]

where \( C_c \) is the carrying cost per unit per year, \( C_o \) is the cost per order, \( Q \) is the order size and \( D \) is the annual demand.

Therefore, in this case the optimal order quantity \( Q_{\text{optimal}} \) is given by

\[
Q_{\text{optimal}} = \sqrt{\frac{2DC_o}{C_c}}.
\]

Fuzzy Economic Order Quantity Model

In the EOQ model we now take the parameters \( C_o, C_c, \) and \( D \) as fuzzy numbers, and denote the fuzzy ordering cost by \( \tilde{C}_o \), the fuzzy annual demand \( \tilde{D} \) and the fuzzy carrying cost by \( \tilde{C}_c \). Let \( C_{0} (\alpha), C_{c} (\alpha), \) and \( D (\alpha) \) denote the \( \alpha \)-cuts of \( \tilde{C}_o, \tilde{C}_c, \) and \( \tilde{D} \) respectively. Thus, for \( 0 \leq \alpha \leq 1 \) we have

\[
C_0 (\alpha) = [C_{00}(\alpha), C_{01}(\alpha)] \quad (2.4)
\]
\[
C_c (\alpha) = [C_{c0}(\alpha), C_{c1}(\alpha)] \quad (2.5)
\]
\[
D (\alpha) = [D_1(\alpha), D_2(\alpha)] \quad (2.6)
\]

Analogous to (2.3), we have

\[
TC (\alpha) = \left[ C_{o1}(\alpha), C_{o2}(\alpha) \right] \left[ D_1(\alpha), D_2(\alpha) \right] \frac{D_1(\alpha), D_2(\alpha)}{Q} + \left[ C_{c1}(\alpha), C_{c2}(\alpha) \right] \frac{Q}{2}
\]
\[
= \left[ \left( C_{o1}(\alpha) D_1(\alpha) \right) + \left( \frac{C_{c1}(\alpha)Q}{2} \right), \left( C_{o2}(\alpha) D_2(\alpha) \right) + \left( \frac{C_{c2}(\alpha)Q}{2} \right) \right] \quad (2.7)
\]

Let \( TC_1 (\alpha) \) and \( TC_2 (\alpha) \) for \( 0 \leq \alpha \leq 1 \) denote the lower bound and the upper bound, respectively, of the \( \alpha \)-cuts for fuzzy total cost. Then,

\[
TC_1 (\alpha) = \left( \frac{C_{o1}(\alpha) D_1(\alpha)}{Q} + \frac{C_{c1}(\alpha)Q}{2} \right) \quad (2.8)
\]
\[
TC_2 (\alpha) = \left( \frac{C_{o2}(\alpha) D_2(\alpha)}{Q} + \frac{C_{c2}(\alpha)Q}{2} \right) \quad (2.9)
\]
The lower possibilistic mean and upper possibilistic mean for fuzzy total cost are as follows.

\[
E_L(TC) = 2 \int_0^1 \alpha \left[ TC_1(\alpha) \right] d\alpha = 2 \int_0^1 \alpha \left( \frac{C_{o1}(\alpha)D_1(\alpha)}{Q} + \frac{C_{c1}(\alpha)Q}{2} \right) d\alpha \tag{2.10}
\]

\[
E_R(TC) = 2 \int_0^1 \alpha \left[ TC_1(\alpha) \right] d\alpha = 2 \int_0^1 \alpha \left( \frac{C_{o2}(\alpha)D_2(\alpha)}{Q} + \frac{C_{c2}(\alpha)Q}{2} \right) d\alpha \tag{2.11}
\]

Now, the possibilistic mean for fuzzy total cost is

\[
E(TC) = \int_0^1 \alpha \left[ TC_1(\alpha) + TC_2(\alpha) \right] d\alpha = \int_0^1 \alpha \left( \frac{C_{o1}(\alpha)D_1(\alpha)}{Q} + \frac{C_{c1}(\alpha)Q}{2} + \frac{C_{o2}(\alpha)D_2(\alpha)}{Q} + \frac{C_{c2}(\alpha)Q}{2} \right) d\alpha \tag{2.12}
\]

The interval valued possibilistic mean \(IPM(TC)\) is

\[
IPM(TC) = \left[ 2 \int_0^1 \alpha \left[ TC_1(\alpha) \right] d\alpha, \ 2 \int_0^1 \alpha \left[ TC_2(\alpha) \right] d\alpha \right] = \left[ 2 \int_0^1 \alpha \left( \frac{C_{o1}(\alpha)D_1(\alpha)}{Q} + \frac{C_{c1}(\alpha)Q}{2} \right) d\alpha, \ 2 \int_0^1 \alpha \left( \frac{C_{o2}(\alpha)D_2(\alpha)}{Q} + \frac{C_{c2}(\alpha)Q}{2} \right) d\alpha \right] \tag{2.13}
\]

(a) In order to find the lower possibilistic optimal economic order quantity, we set

\[
\frac{d}{dQ} \left( 2 \int_0^1 \alpha \left( \frac{C_{o1}(\alpha)D_1(\alpha)}{Q} + \frac{C_{c1}(\alpha)Q}{2} \right) d\alpha \right) = 0 \tag{2.14}
\]

and then solve for \(Q\). That value of \(Q\) that minimize (2.14) is the lower possibilistic order quantity \((Q = EOQ_\ast)\).

(b) In order to find the upper possibilistic optimal economic order quantity, we set

\[
\frac{d}{dQ} \left( 2 \int_0^1 \alpha \left( \frac{C_{o2}(\alpha)D_2(\alpha)}{Q} + \frac{C_{c2}(\alpha)Q}{2} \right) d\alpha \right) = 0 \tag{2.15}
\]

and then solve for \(Q\). That value of \(Q\) that minimize(2.15) is the upper possibilistic order quantity \((Q = EOQ^\ast)\).

(c) In order to find the possibilistic optimal economic order quantity, we set

\[
\frac{d}{dQ} \left( \int_0^1 \alpha \left( \left( \frac{C_{o1}(\alpha)D_1(\alpha)}{Q} + \frac{C_{c1}(\alpha)Q}{2} \right) + \left( \frac{C_{o2}(\alpha)D_2(\alpha)}{Q} + \frac{C_{c2}(\alpha)Q}{2} \right) \right) \right) = 0 \tag{2.16}
\]

and then solve for \(Q\). That value of \(Q\) that minimize (2.16) is the possibilistic order quantity \((Q = EOQ)\).
Remark 2.1 It is important to remark here that in this case $EOQ \neq \frac{EOQ_1 + EOQ^*}{2}$.

Theorem 2.1

For $0 \leq \alpha \leq 1$, let the fuzzy ordering cost $C_{0}(\alpha)$, fuzzy Demand $D(\alpha)$ and the fuzzy holding cost $C_{c}(\alpha)$ be given by

\begin{align*}
C_{0}(\alpha) &= \left[ C_{01}(\alpha), C_{02}(\alpha) \right] = \left[ c_1 + \alpha (c_2 - c_1), c_4 + \alpha (c_3 - c_4) \right] \quad (2.17) \\
D(\alpha) &= \left[ D_{1}(\alpha), D_{2}(\alpha) \right] = \left[ D_1 + \alpha (D_2 - D_1), D_4 + \alpha (D_3 - D_4) \right] \quad (2.18) \\
C_{c}(\alpha) &= \left[ C_{c1}(\alpha), C_{c2}(\alpha) \right] = \left[ h_1 + \alpha (h_2 - h_1), h_4 + \alpha (h_3 - h_4) \right] \quad (2.19)
\end{align*}

respectively. Then, the fuzzy total cost $TC(\alpha)$ in terms of the model parameters and the $\alpha$-level sets is given by

\begin{align*}
TC(\alpha) &= \left[ TC_1(\alpha), TC_2(\alpha) \right] \\
TC_1(\alpha) &= \left( \frac{(c_1 + \alpha (c_2 - c_1)) (D_1 + \alpha (D_2 - D_1))}{Q} \right) + \left( \frac{(h_1 + \alpha (h_2 - h_1)) Q}{2} \right) \\
TC_2(\alpha) &= \left( \frac{(c_4 + \alpha (c_3 - c_4)) (D_4 + \alpha (D_3 - D_4))}{Q} \right) + \left( \frac{(h_4 + \alpha (h_3 - h_4)) Q}{2} \right)
\end{align*}

and

\begin{align*}
E_L(TC) &= \frac{1}{6} \left( \frac{3c_2D_2 + D_1c_2 + c_1D_2 + c_1D_1 + 2Q^2h_2 + h_1Q^2}{Q} \right) \quad (2.20) \\
E_R(TC) &= \frac{1}{6} \left( \frac{3c_3D_3 + D_4c_3 + c_4D_3 + c_4D_4 + 2Q^2h_3 + h_4Q^2}{Q} \right) \quad (2.21)
\end{align*}

and

\begin{align*}
E(TC) &= \frac{1}{12} \left( \frac{3c_2D_2 + D_1c_2 + c_1D_2 + c_1D_1 + 3c_3D_3 + D_4c_3 + c_4D_3 + c_4D_4 + 2Q^2h_2 + h_1Q^2 + 2Q^2h_3 + h_4Q^2}{Q} \right) \quad (2.22)
\end{align*}

In what follows, we will always refer to $E_L(TC)$, $E_R(TC)$ and $E(TC)$ as fuzzy possibilistic costs (total inventory cost). The proof for $E_L(TC)$, $E_R(TC)$ and $E(TC)$ follows from definitions given in Section 1.2 and therefore is omitted.

In the Theorem below, we will discuss a possibilistic approach to find the economic order quantity in a possibilistic setup. Note that $EOQ_1$ is the fuzzy possibilistic economic order quantity using expression (2.20) $EOQ^*$ is the fuzzy possibilistic economic order quantity using expression (2.21) and $EOQ$ is the fuzzy possibilistic economic order quantity using expression (2.22) respectively.
Theorem 2.2

Using expression for \(E_L(TC), E_R(TC)\) and \(E(TC)\) from Theorem 2.1, we have the following values for lower \(EOQ_*, EOQ^*\) and \(EOQ\) respectively.

\[
EOQ_* = \frac{\sqrt{((2h_2 + h_1) (c_1 D_2 + c_1 D_1 + 3c_2 D_2 + D_1 c_2))}}{2h_2 + h_1} \tag{2.23}
\]

\[
EOQ^* = \frac{\sqrt{((2h_3 + h_4) (c_4 D_3 + c_4 D_4 + 3c_5 D_3 + D_4 c_3))}}{2h_3 + h_4} \tag{2.24}
\]

\[
EOQ = \frac{\sqrt{((2h_2 + h_1 + 2h_3 + h_4) (3c_2 D_2 + D_1 c_2 + c_1 D_2 + c_1 D_1 + 3c_3 D_3 + D_4 c_3 + c_4 D_3 + c_4 D_4))}}{2h_2 + h_1 + 2h_3 + h_4} \tag{2.25}
\]

and total cost at \(EOQ_*\) is

\[
TC_L(Q = EOQ_*) = \frac{(3c_2 D_2 + D_1 c_2 + c_1 D_2 + c_1 D_1) (2h_2 + h_1)}{6} \frac{((2h_2 + h_1) (c_1 D_2 + c_1 D_1 + 3c_2 D_2 + D_1 c_2))}{\sqrt{((2h_2 + h_1) (c_1 D_2 + c_1 D_1 + 3c_2 D_2 + D_1 c_2))}} + \frac{((2h_3 + h_4) (c_4 D_3 + c_4 D_4 + 3c_5 D_3 + D_4 c_3))}{6} \tag{2.26}
\]

The total cost at \(EOQ^*\) is given by

\[
TC_R (Q = EOQ^*) = \frac{(3c_3 D_3 + D_4 c_3 + c_4 D_3 + c_4 D_4) (2h_3 + h_4)}{6} \frac{((2h_3 + h_4) (c_4 D_3 + c_4 D_4 + 3c_5 D_3 + D_4 c_3))}{\sqrt{((2h_3 + h_4) (c_4 D_3 + c_4 D_4 + 3c_5 D_3 + D_4 c_3))}} + \frac{((2h_3 + h_4) (c_4 D_3 + c_4 D_4 + 3c_5 D_3 + D_4 c_3))}{6} \tag{2.27}
\]

and the total cost at \(EOQ\) is

\[
= \frac{(3c_2 D_2 + D_1 c_2 + c_1 D_2 + c_1 D_1 + 3c_3 D_3 + D_4 c_3 + c_4 D_3 + c_4 D_4)}{12} \frac{(2h_2 + h_1 + 2h_3 + h_4)}{\sqrt{((2h_2 + h_1 + 2h_3 + h_4) (3c_2 D_2 + D_1 c_2 + c_1 D_2 + c_1 D_1 + 3c_3 D_3 + D_4 c_3 + c_4 D_3 + c_4 D_4))}} + \frac{(2h_2 + h_1 + 2h_3 + h_4)}{12} \frac{((2h_2 + h_1 + 2h_3 + h_4) (3c_2 D_2 + D_1 c_2 + c_1 D_2 + c_1 D_1 + 3c_3 D_3 + D_4 c_3 + c_4 D_3 + c_4 D_4))}{(2h_2 + h_1 + 2h_3 + h_4)} \tag{2.28}
\]

We will established the proof for (2.23) and (2.26) only. The proof for (2.24), (2.25), (2.27) and (2.28) follow along similar lines.

**Proof:** Since,

\[
E_L(TC) = \frac{1}{6} \frac{3c_2 D_2 + D_1 c_2 + c_1 D_2 + c_1 D_1 + 2Q^2 h_2 + h_4 Q^2}{Q}
\]
To find the optimal $EOQ_*$ we solve $\frac{d}{dQ} E_L(TC) = 0$, thus
$$\frac{1}{6} \frac{2Q^2 h_2 + h_1 Q^2 - 3c_2 D_2 - D_1 c_2 - c_1 D_2 - c_1 D_1}{Q^2} = 0$$
Solving for $Q$, we have
$$Q = \pm \frac{1}{2h_2 + h_1} \sqrt{\left(2h_2 + h_1\right) \left(c_1 D_2 + c_1 D_1 + 3c_2 D_2 + D_1 c_2\right)}$$
(2.29)

It is important to note here that in (2.29) we only consider the positive value of $Q$, since $Q$ cannot be negative. The value of $Q$ that optimizes $E_L(TC)$ will be referred to as $EOQ_*$. Since,
$$\frac{d^2 E_L(TC)}{dQ^2} = \frac{1}{3} \frac{c_1 D_2 + c_1 D_1 + 3c_2 D_2 + D_1 c_2}{Q^3} > 0,$$
thus we have minimum value at $EOQ_*$. Thus, (2.23) is proved.

The result for expression (2.26) follows directly by evaluating $E_L(TC)$ at $EOQ_*$ and is as follows,
$$E_L(TC = EOQ_*) = \frac{1}{6} \frac{3c_2 D_2 + D_1 c_2 + c_1 D_2 + c_1 D_1 + 2Q^2 h_2 + h_1 Q^2}{Q}$$
$$= \frac{\left(3c_2 D_2 + D_1 c_2 + c_1 D_2 + c_1 D_1\right) \left(2h_2 + h_1\right)}{6\sqrt{\left(2h_2 + h_1\right) \left(c_1 D_2 + c_1 D_1 + 3c_2 D_2 + D_1 c_2\right)}} + \frac{\sqrt{\left(2h_2 + h_1\right) \left(c_1 D_2 + c_1 D_1 + 3c_2 D_2 + D_1 c_2\right)}}{6}$$

This proves the results (2.26).

**Conclusion**

In this paper we have developed possibilistic approach to the economic order quantity inventory model. The methodology proposed in this paper may also be applicable to other inventory models within a supply chain environment.
References


