Product storage facilities play an important role to provide appropriate customer service levels, maintain product availability and perform value-adding activities in a supply chain. Two main warehouse options available to the management are public and private warehousing. The major decision involved in these situations is to decide how much warehouse space to own and how much to outsource to public warehouses. The present paper addresses this important strategic level decision problem using a linear programming model. The paper also proposes a solution algorithm and explains the model and solution methodology with a numerical illustration.

Introduction

Product storage function represents a node on the supply chain network and therefore, it is an important part of supply chain management process. Towards this end, storage facilities play a key role to ensure customer service levels by stocking products and materials before, during and after the assembly operations. In addition, these facilities can also be used to perform a wide range of value-adding services in the supply chain. Therefore, the importance of storage facilities in a supply chain can not be over-emphasized. Besides providing customer service level and value-adding benefits, storage function allows economies of scale in transportation, maintains an uninterrupted supply for critical operations, handles the market fluctuations and allows firms to take advantage of quantity discounts in material acquisitions. Another important benefit of storage function is to overcome the time and place gap between production and consumption points, thereby adding time and place value so critical in the supply chains.

In general, there are two main options available to the management in terms of public warehousing and private warehousing, although there could be a whole range of hybrid options available based on a suitable mix between these two options. Public warehouses are owned and operated by a third party; they are meant for short-term usage and customers are charged on a per square (or cubic) footage space needed over a certain period of time. The biggest advantage of a public warehouse accrues from the fact that it requires no fixed capital investment in terms of real estate purchase, material handling, equipment or the warehouse training costs. Most of the cost to a customer is a variable cost for warehouse usage. These types of warehouses are particularly attractive to companies that don't want to live with the liability of a long-term commitment and the prohibitive cost of a warehouse change-over. This includes the companies that are penetrating into newer markets; companies that prefer greater flexibility in switching from one market segment to another and companies that have a seasonality factor built into their demand and supply side. From an accounting perspective, storage payments strictly based on volumetric measures help to accurately predict the logistics costs and hence the product costs. From an administrative perspective, this warehouse option involves minimal labor dispute and union conflicts; and therefore results in considerable savings in terms of management effort. With a substantial portion of product storage
activities handled in public warehouses, volume and consolidation factors makes it possible for warehousing experts to operate in a cost effective manner. Additionally, at a reasonable cost, the customer company can access specialized features such as packaging, break-bulk, temperature controlled storage, security, dedicated bays, special handling of fragile parts etc. On the downside, public warehouses mainly provide localized services and operate at a smaller scale. Therefore, they may not have enough space availability, IT infrastructure or data transfer capabilities needed by the bigger companies to integrate their supply chains.

On the other hand, private warehousing (also known as proprietary warehousing) is usually managed as a business entity within the same company. It requires a substantial corporate investment in land, building, equipment and warehouse employees. Private warehousing option allows a firm greater control over inventory management, material flows and space utilization. It provides product visibility while the product is in storage, therefore, making it easier to integrate the storage function with the other functions of company's logistics network design. Private facilities can be expanded or renovated to allow the storage of products after a major design change. In addition, the company keeps on reaping the benefits of real estate appreciation while getting the depreciation tax shelters on buildings and equipment. Furthermore, channelizing the product through its private warehouses, a firm projects the image of stability and permanence with lasting operations. This factor accords well with future partners in the supply chain. The major drawback is the substantial investment in capital and equipment needed to operate the supply chain. Larger facilities stay under-utilized during a period of low demand. Many firms seek to alleviate this drawback by renting out under-utilized space but at the cost of loosing valuable flexibility to respond to sudden surge in market demands and changing customer preferences. Warehouse ownership is a risky proposition and there is always a chance that the firm may experience difficulties in selling it at a later stage due to factors such as warehouse customization and recession in the real estate market. This limits the agility of a company to pursue newer business opportunities and keeps its capital tied up in an unnecessary venture.

As evident from this discussion, there are pros and cons of both private and public warehousing. When the rates of returns on real estate investments become comparable with other investment opportunities, a viable approach for many companies is to use a combination of private and public warehousing. Such an approach uses a minimal private warehousing option to store stable and uniform inventory levels while any demand fluctuation and uncertainty is outsourced to public warehouses. Then the crux of the problem is to decide how much warehouse space to own and how much to outsource. The present paper endeavors to address this important strategic level decision problem.

Literature Survey

Traditionally, the warehousing sizing problems have been dealt with a storage capacity and customer service perspective where the total cost is minimized while maintaining a desired customer service level. A review of such optimization model is available in Cormier and Gunn (1996). Cormier and Gunn (1996) also modeled the private versus leased space trade-off under the assumption of constant product demand. One finding of this work is the realization that leasing is significantly more beneficial when the warehouse size is tightly constrained. The dynamic version of warehouse capacity models as an extension of the plant capacity model was considered in Manne and Veinott (1967). A review of such capacity expansion models is available in Luss (1982). White and Francis (1971) examined the problem of determining optimum size under conditions of deterministic and probabilistic storage demand. The latter is formulated as a linear programming problem and transformed via duality theory into an equivalent network flow problem for efficient solution. Costs considered in White and Francis (1971) model are warehouse construction costs, storage of products within the facility and storage demand outsourced. Hung and Fisk (1984) also provided a linear programming formulation for finding the economical size of warehouses. The general warehouse design methodology under which the current problem exists as a sub-problem, has been addressed by
Ashayeri and Gelders (1985). Bhaskaran and Malmborg (1990) expressed the economic trade-offs in designing reserve storage areas in warehouses. Giles and Eldon (1992) provided a review of general warehousing models in operations research. Rao and Rao (1998) discussed the economic warehousing sizing problem under seasonal demand. Rao and Rao (1998) presented extensions of the warehousing sizing model including time varying costs, economies of scale in capital expenditure, concave costs, stochastic versions etc. Goh et. al (2001) model considers a warehousing sizing problem whose objective is to minimize the total cost of ordering, holding and warehousing of inventory. The warehouse cost structure examined herein is not the unit rate type, but rather a more realistic step function of the warehouse space to be acquired. Petinis et. al (2005) considered multi-product case as a non-linear warehouse sizing problem with the objective to express the minimizing of inventory ordering and holding costs and solved it using successive quadratic programming procedure. Baker and Canessa (2009) reviewed the literature concerning various steps and the associated tools in warehouse design and this review covers period 1973 to 2006. As evident from the above survey, the issue of warehouse sizing has not been extensively dealt with in the past work. There are very few publications (discussed above) available on the sizing problems in warehouses. Even the general issue of warehouse design has received very little attention in the past literature. Several authors acknowledge the lack of past work in this area through statements such as:

“Very few papers deal with the general warehouse design (Ashayeri and Gelders, 1985)”;
“There is not a procedure for systematically analyzing the requirements and designing a warehouse to meet the operational need using the most economic technology (Rowley, 2000)”;
“A sound theoretical basis for a warehouse design methodology still seems to be lacking (Rouwenhorst et al., 2000).

The present paper considers the warehouse sizing problem when public warehouse option is available. The contribution of the present paper lies in presenting a linear programming formulation and solution algorithm for the warehousing sizing problem and can be used as an alternative to approaches such as Ballou (1974) and Hung and Fisk (1984). The solution algorithm in the present approach is simple and easy to understand and can be effectively used as a classroom tool to introduce warehouse capacity optimization decisions.

The Model

Notation

Let,

\[ T = \text{number of discrete time periods in the warehouse demand planning horizon.} \]
\[ t = \text{time index in the planning horizon. } t=1, 2, \ldots, T. \]
\[ X = \text{private warehouse space ownership (in square feet).} \]
\[ X_t = \text{forecasted warehouse space requirements in time period } t \text{ (in square feet).} \]
\[ K = \text{number of time periods where demand exceeds the private ownership, thereby necessitating rental or outsourcing option.} \]
\[ R_t = \text{warehouse space rented in time period } t. \]
\[ R_t = \begin{cases} 
X_t - X, & \text{for } X_t \geq X, \\
0, & \text{otherwise} 
\end{cases} \quad \text{for } t = 1, 2, \ldots, T. \]
\[ P_t = \text{private warehouse space under-utilization in time period } t. \]
\[ P_t = \begin{cases} 
X - X_t, & \text{for } X_t \leq X, \\
0, & \text{otherwise} 
\end{cases} \quad \text{for } t = 1, 2, \ldots, T. \]
\[ O_t = \text{operational utilization level of private warehouse in time period } t. \]

\[ f = \text{fixed cost of warehouse ownership per square foot per unit of time. Such an estimate can be conveniently obtained from real estate parametric estimation over the useful life of warehouse.} \]

\[ v = \text{variable operational cost of private warehouse per square foot per unit of time.} \]

\[ r = \text{variable rental cost of public warehouse per square foot per unit of time.} \]

\[ Y_t = \text{a binary variable which takes a value of ‘1’ when rental space is required in time period } t \text{ and it takes a null value when no rental space is needed.} \]

\[ Y_t = \begin{cases} 1, & \text{for } X_t \geq X \\ 0, & \text{otherwise} \end{cases}, \text{ for } t = 1,2, \ldots, T. \]

**Linear Programming Formulation**

The linear program for this problem can be expressed as follows.

Minimize \( Z = f \cdot X \cdot T + v \cdot \sum_{t=1}^{T} O_t + r \cdot \sum_{t=1}^{T} R_t \) \hspace{1cm} (1)

This model represents the minimization of three cost components: the fixed cost of private warehouse, the variable operational cost of a private warehouse and the variable rental cost of a public warehouse, subject to the following constraints:

\[ X + R_t - P_t = X_t, \text{ for } t = 1,2,\ldots,T. \] \hspace{1cm} (2)

\[ R_t \leq M \cdot Y_t, \text{ for } t = 1,2,\ldots,T. \] \hspace{1cm} (3)

\[ P_t \leq M \cdot (1 - Y_t), \text{ for } t = 1,2,\ldots,T. \] \hspace{1cm} (4)

\[ X - P_t - O_t = 0, \text{ for } t = 1,2,\ldots,T. \] \hspace{1cm} (5)

\( O_t, R_t, P_t, X \geq 0; \ Y_t \text{ : binary.} \)

Constraints set (2) is a space balance constraint that ensures the total space requirements is the sum of warehouse space owned, plus warehouse house space rented minus any underutilization of space ownership. It may be noted that when rental space is needed (i.e. when \( X_t \geq X \) or when \( R_t \geq 0 \) in the time period \( t \), the private warehouse underutilization level is zero (i.e. \( P_t = 0 \)). On the other hand, when we experience private warehouse underutilization (i.e. \( X_t \leq X \) or \( P_t \geq 0 \)), the corresponding warehouse rental space must be zero (\( R_t = 0 \)). Constraint sets (3) and (4) essentially model these if-then types of situations. Parameter \( M \) in constraints (3) and (4) represents a very large number. The value of \( M \) should be chosen such that it is greater than the peak warehouse requirements i.e. \( M \geq \max(X) \). Constraint set (5) is another space balance constraint for private warehouse space that ensures that operational level of private warehouse is equal to the space ownership level minus the space under-utilization in that time period.
The above linear program can be conveniently solved for smaller size problems, but since the strategic ownership decisions are long term decisions, they involve longer planning horizons which can significantly increase the size of the problem. Even for a small sized problem consisting of 12 months (periods) in a year, the problem consists of 48 constraints. The problem size in terms of number of constraints increases linearly w.r.t. number of time periods. It may also be noted that granularity and length of planning horizon normally follows the granularity and planning horizon used for the warehouse space forecast. Normally, finer the granularity of time buckets used in the forecast (i.e. using weeks or days instead of months), more accurate will be the results, however, it will significantly increase the size of the problem in terms of number of constraints. The problem size can be somewhat reduced by bringing constraint (5) into the objective function. However, we suggest an optimal solution procedure to solve large sized problems.

Using the notation given in earlier section, the problem can be represented as the minimization of total cost function, TC, given below:

\[ TC = f . X . T + v . \sum_{t=1}^{T} \min(X_t, X) + r . \sum_{t:X_t>X} (X_t - X) \]  

(6)

The summation operators in equation (6) can be expanded to give equation (7) as follows.

\[ TC = f . X . T + v . \sum_{t:X_t>X} X + v . \sum_{t:X_t<X} (X_t) + r . \sum_{t:X_t>X} (X_t) - r . \sum_{t:X_t>X} X \]  

(7)

As per the notation, warehouse demand exceeds the private space ownership during \( K \) time buckets, therefore, the above equation can be modified as follows.

\[ TC = f . X . T + v . \sum_{t:X_t<X} (X_t) + r . \sum_{t:X_t>X} (X_t) - r . X . K \]  

(8)

Since the total cost function is convex over \( X \), an optimal point can be obtained by setting the partial derivative w.r.t. \( X \) equal to zero.

\[ \frac{\partial (TC)}{\partial X} = 0 \Rightarrow f . T + v . K - r . K = 0 \]

(9)

Therefore, the optimal warehouse space ownership should be chosen such that the optimality condition given in relationship (9) is satisfied.

It may further be noted that when the per square foot rental cost (\( r \)) is lower than the ownership costs (\( f + v \)), the common sense solution is to rent all the space during all the periods. Therefore, we discuss the model for those situations when \( r \geq (f + v) \).
Case 1. When \( r = f + v \), this type of cost structure gives \( K \) equal to \( T \) as per optimality condition (9). This calls for outsourcing the demand fluctuations in all the time periods to public warehouses as such a solution can not afford any warehouse under-utilization. Therefore, the optimal strategy in this case requires setting the private warehousing space to a certain minimal stable demand level, i.e. \( X \leq \min (X_t) \).

Case 2. When \( r > f + v \), this cost structure implies \( K \) less than \( T \) as per optimality condition (9). This optimality strategy requires a portion of demand fluctuations in certain time periods to be handled through private warehousing and the rest is outsourced to public warehouses. An optimal equilibrium point exists where the total cost is minimized. If the owned space is increased beyond this optimal point, incremental cost of facility under-utilization forces it back towards equilibrium. On the other hand, if warehouse ownership is reduced from this optimal point, the marginal cost savings of private warehouse as compared with the rental option, again pushes it towards optimal point. Such an optimal value of warehouse ownership lies between \( \min (X_t) \leq X \leq \max (X_t) \). The exact value of this optimal warehouse ownership space corresponds with the optimal point given in condition (9). Below, we provide a sorting procedure to obtain this optimal value of space ownership.

Identifying Optimal Solution

Step 1. Obtain the optimal value of \( K \) using formula (9).
Step 2. Sort the entire data set, \( X_t \) in the descending order.
Step 3. Find the \( X_t \) value at \( K+1^{th} \) position in the sorted data set.
Step 4. Set optimal value of \( X \) equal to the \( X_t \) value identified in step 3.
Step 5. Compute the rental level in month \( t \), i.e. \( R_t = \max (0, X_t - X) \).

\[
\text{Compute } \sum_{t=1}^{T} R_t .
\]

Step 6. Compute operational level of private warehouse in month \( t \), i.e. \( O_t = \min (X, X_t) \).

\[
\text{Compute } \sum_{t=1}^{T} O_t .
\]

Step 7. Compute the total optimal cost, \( TC = f \cdot X \cdot T + v \cdot \sum_{t=1}^{T} O_t + r \cdot \sum_{t=1}^{T} R_t . \)

Illustrative Example

In this illustrative example, we consider a public warehouse rental cost of $20 per square feet per month whereas private warehouse costs are $10 per square feet per month as the fixed cost and a variable operational cost of $4 per square feet per month. The length of planning horizon is 12 months. As discussed earlier, the length of planning horizon and granularity of time buckets reflect on the problem size and accuracy of results. In this example, we assume that storage requirements are forecasted on a monthly basis and public warehouse space rental is also available on a monthly basis, the storage requirements over the next 12 months are given in Table 1 as follows:
We use the value of \( M = 100,000 \) which is larger than the peak requirement of 10000. Using the model given in equations (1-5), the linear programming formulation for the above example can be written as follows:

\[
\text{Minimize } Z = 120X + 4(O_1 + O_2 + \ldots + O_{12}) + 20(R_1 + R_2 + \ldots + R_{12})
\]

Subject to:

\[
X + R_t - P_t = X_t, \text{ for } t = 1,2,\ldots,12.
\]

\[
R_t \leq M \cdot Y_t, \text{ for } t = 1,2,\ldots,12.
\]

\[
P_t \leq M \cdot (1 - Y_t), \text{ for } t = 1,2,\ldots,12.
\]

\[
X - P_t - O_t = 0, \text{ for } t = 1,2,\ldots,12.
\]

\[
P_t, O_t, R_t, X \geq 0; \ Y_t = 0/1
\]

We solve the above mixed 0-1 linear programming formulation using excel solver to obtain the following solution given in Table 2.

### Table 1

<table>
<thead>
<tr>
<th>Month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X_t) (Sq.ft)</td>
<td>5000</td>
<td>7000</td>
<td>5500</td>
<td>7500</td>
<td>8000</td>
<td>6000</td>
<td>7000</td>
<td>9500</td>
<td>10000</td>
<td>8000</td>
<td>7000</td>
<td>6500</td>
</tr>
</tbody>
</table>

### Table 2

<table>
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<th>(O_6)</th>
<th>(O_7)</th>
<th>(O_8)</th>
<th>(O_9)</th>
<th>(O_{10})</th>
<th>(O_{11})</th>
<th>(O_{12})</th>
</tr>
</thead>
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<td>(R_3)</td>
<td>(R_4)</td>
<td>(R_5)</td>
<td>(R_6)</td>
<td>(R_7)</td>
<td>(R_8)</td>
<td>(R_9)</td>
<td>(R_{10})</td>
<td>(R_{11})</td>
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<td>1000</td>
<td>0</td>
<td>0</td>
<td>2500</td>
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<td>1000</td>
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<td>0</td>
</tr>
<tr>
<td>Var</td>
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<td>(P_2)</td>
<td>(P_3)</td>
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<td>(P_5)</td>
<td>(P_6)</td>
<td>(P_7)</td>
<td>(P_8)</td>
<td>(P_9)</td>
<td>(P_{10})</td>
<td>(P_{11})</td>
<td>(P_{12})</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>500</td>
</tr>
<tr>
<td>Var</td>
<td>(Y_1)</td>
<td>(Y_2)</td>
<td>(Y_3)</td>
<td>(Y_4)</td>
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<td>(Y_9)</td>
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<td>(Y_{11})</td>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Solution using Optimal Procedure

We demonstrate the application of optimal solution procedure as follows.

Step 1. The optimal value of $K$ using formula (9) has been found to be $K = 7.5$ which is rounded up to 8.

Step 2. The data set given in Table 1 was sorted in the descending order.

Step 3. We identify the $X_t$ value at 8th position in the sorted data set. The value at this position is $X_t = 7000$.

Step 4. We set optimal value of $X = 7000$ sq. feet.

Step 5. We compute the rental level in month $t$, i.e.

\[
R_1 = \max (0, X_1 - X) = \max (0, 5000 - 7000) = 0 \\
R_2 = \max (0, X_2 - X) = \max (0, 7000 - 7000) = 0 \\
R_3 = \max (0, X_3 - X) = \max (0, 5500 - 7000) = 0 \\
R_4 = \max (0, X_4 - X) = \max (0, 7500 - 7000) = 500 \\
R_5 = \max (0, X_5 - X) = \max (0, 8000 - 7000) = 1000 \\
R_6 = \max (0, X_6 - X) = \max (0, 6000 - 7000) = 0 \\
R_7 = \max (0, X_7 - X) = \max (0, 7000 - 7000) = 0 \\
R_8 = \max (0, X_8 - X) = \max (0, 9500 - 7000) = 2500 \\
R_9 = \max (0, X_9 - X) = \max (0, 10000 - 7000) = 3000 \\
R_{10} = \max (0, X_{10} - X) = \max (0, 8000 - 7000) = 1000 \\
R_{11} = \max (0, X_{11} - X) = \max (0, 7000 - 7000) = 0 \\
R_{12} = \max (0, X_{12} - X) = \max (0, 6500 - 7000) = 0
\]

\[\sum_{t=1}^{12} R_t = 8000\] sq. feet.

Step 6. We compute the operational level of private warehouse in month $t$, i.e.

\[
O_1 = \min (X, X_1) = \min (7000, 5000) = 5000 \\
O_2 = \min (X, X_2) = \min (7000, 7000) = 7000 \\
O_3 = \min (X, X_3) = \min (7000, 5500) = 5500 \\
O_4 = \min (X, X_4) = \min (7000, 7500) = 7000 \\
O_5 = \min (X, X_5) = \min (7000, 8000) = 7000 \\
O_6 = \min (X, X_6) = \min (7000, 6000) = 6000 \\
O_7 = \min (X, X_7) = \min (7000, 7000) = 7000 \\
O_8 = \min (X, X_8) = \min (7000, 9500) = 7000 \\
O_9 = \min (X, X_9) = \min (7000, 10000) = 7000 \\
O_{10} = \min (X, X_{10}) = \min (7000, 8000) = 7000 \\
O_{11} = \min (X, X_{11}) = \min (7000, 7000) = 7000 \\
O_{12} = \min (X, X_{12}) = \min (7000, 6500) = 6500
\]

\[\sum_{t=1}^{12} O_t = 79000\] sq. feet.

Step 7. The total optimal cost, $TC = 120(7000) + 4(79000) + 20(8000)$

\[= 81,316,000.00\]

The above solution procedure has been modeled in Microsoft Excel and can be conveniently used to solve large sized problems in an optimal fashion.
Conclusion and Extensions

The present paper addresses the warehousing option decision in a supply chain. Predominantly, there are two main options available to the management in terms of public warehousing and private warehousing. However, the optimal strategy is often a mix of these two options by maintaining a certain level of warehouse ownership and outsourcing any fluctuations to public warehouses. The main decision, therefore, is to decide how much space to own. The present paper modeled this decision problem as a linear program and solved it using an optimal solution algorithm. The approach is convenient to learn and can be effectively used as a classroom tool to teach warehouse sizing and capacity optimization. In order to establish its computational superiority over other approaches to solve practical problems, a computational comparison between various approaches is needed. This is one direction for future work. Several other extensions of the model are also possible. The paper mainly deals with a static warehousing problem which maintains a constant private storage space. One possible extension of this work is to modify the model for dynamic situations where private warehouse space is allowed to vary from month to month. Such a scenario includes cases where additional outside space may be used during summer months, opening or closing storage areas based on warehouse personnel or equipment availability.
References


