MULTIPLE CRITERIA AGGREGATION PROCEDURE FOR MIXED EVALUATIONS INCLUDING THE WEAK PREFERENCE RELATION

A multiple criteria aggregation procedure for mixed evaluations is designed to deal with different kinds of information imperfections. It leads to a global preference relational system containing the strict preference, the weak preference, the indifference and the incomparability relations. Introducing the weak preference relation allows for more nuance and flexibility to the decision-maker in this context.

Introduction

Many multiple criteria decision situations imply the presence of different types of information imperfections at the same time (Ben Amor, 2004). However, most of the existing discrete multiple criteria methods, which consider the information imperfections, treat only one type of imperfection at the time. In fact, the majority of these procedures are based on a probabilistic or a fuzzy modeling. The multiple criteria methods accepting mixed evaluations (punctual, stochastic, fuzzy or else) are rather rare. We find in the literature at least two: NAIADE (Munda, 1995) and PAMSSEM (Martel et al., 1997; Guitouni et al., 2008).

Multiple criteria methods are based on the \((A, X, E)\) model where \(A\) is a set of potential alternatives, \(X\) is a finite set of attributes (or criteria) and \(E\) is a performances table or an evaluation matrix. Some methods associate to each attribute/criterion \(j\) a coefficient of relative importance (or weight) \(w_j\), where \(\sum_{j=1}^{n} w_j = 1\). We have:

\[
A = \{a_1, \ldots, a_i, \ldots, a_m\} \tag{1}
\]

\[
X = \{X_1, \ldots, X_j, \ldots, X_n\} \tag{2}
\]
It should be observed that the information imperfection would notably appear in the evaluations $e_{ij}$.

The aim of the NAIADE and PAMSSEM methods is to establish binary relations between the different pairs of alternatives according to each criterion. These relations are then aggregated to produce "global" relations between the pairs of alternatives, which are exploited afterwards.

Since the evaluations are mixed (different natures of the imperfections), the aggregation of the valued local relations for all the pairs of alternatives is sometimes difficult. In fact, the signification of the numerical values at the local level could be unclear because of the heterogeneity of the measurement scales and the natures of the evaluations. Consequently, conciliating the results of the pair comparisons according to the criteria could be difficult. This observation led us to use all available information to establish "crisp" relations locally between all the pairs of alternatives, then, to aggregate them. Thus, a multiple criteria aggregation procedure for mixed evaluations was designed. It leads to non-valued global preference relations where the p.r.s. (preference relational system) consists of the strict preference, the indifference and the incomparability relations (Ben Amor et al., 2007). This procedure will be extended here to include the weak preference relation in its p.r.s. In fact, enriching the preference relational system will give more freedom to the decision-maker and allow more nuance in the preference modeling. The weak preference relation ($Q$) is an intermediate relation between preference and indifference. For example, it has been introduced by Roy (1978) with the ELECTRE III method. It represents a zone of ambiguity and/or uncertainty where it is difficult to make a distinction between preference and indifference.

In the beginning of this paper, a general modeling framework will be presented to clarify the context in which the proposed procedure is applied (section 2). Our multiple criteria procedure for mixed evaluations comprises two main phases. The first phase (section 3) consists of constructing the local preference relations. This construction is based on an extension of the stochastic dominance concept to the mixed evaluations context; the stochastic dominance providing a uniform treatment in this context. The second phase (section 4) consists of aggregating these local binary relations into global binary relations using an algorithm developed in a group decision context and adapted to the multicriteria aggregation (Jabeur, 2004; Jabeur et al., 2004). The algorithm is here extended to handle the weak preference relation. Finally, the exploitation of the resulting global preference relations is discussed in section 5 using adapted procedures from the work of Jabeur (2004). Conclusions and future work follow in section 6.
General Modeling Framework

The adopted framework accepts, in the uncertain decision-making situations, stochastic, possibilistic or "evidential" evaluations. It also includes fuzzy or ordinal evaluations. We think that among all uncertainty-modeling languages (evidence theory, possibility theory and probability theory), the evidence theory presents the more general framework where possibilities and probabilities are proposed as particular cases. We exploit this property to settle our model.

The performance matrix for an attribute \( j \) is presented in table 1. For each alternative \( a_i \), the evaluation \( e_{ij} \) according to the attribute \( j \) will depend on the set of the nature states \( \Omega^j = \{ \omega^j_1, ..., \omega^j_q, ..., \omega^j_H \} \). The matrix is represented by the values \( x_{ij}^h \) of the set \( X_{ij} = \{ x_{ij}^1, ..., x_{ij}^h, ..., x_{ij}^H \} \). This model assumes also that we are able to indicate the consequence of choosing an alternative \( a_i \) when the element \( \omega^h_i \) (\( h = 1, 2, ..., H \)) is realized, i.e. when \( x_{ij}^h \) is obtained.

Table 1

Performance Matrix for the Attribute \( j \)

<table>
<thead>
<tr>
<th>Alternatives</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>( a_1 )</td>
<td></td>
</tr>
<tr>
<td>( \vdots )</td>
<td></td>
</tr>
<tr>
<td>( a_i )</td>
<td></td>
</tr>
<tr>
<td>( \vdots )</td>
<td></td>
</tr>
<tr>
<td>( a_m )</td>
<td></td>
</tr>
</tbody>
</table>

The prior information available about the nature states (probability functions, belief masses, possibility measures, ...) could be integrated in table 2.

This table is based on a model proposed by the theory of evidence (Shafer, 1976) where the prior information is represented by belief masses associated to the focal elements. These focal elements, which are subsets \( B^j_h \subseteq \Omega^j \) (\( h' = 1, ..., 2^H \)) of the nature states set \( \Omega^j = \{ \omega^j_1, ..., \omega^j_q, ..., \omega^j_H \} \), influence the evaluations of the alternatives according to the attribute \( j \). According to the evidence theory, the evaluation \( e_{ij} \) of an alternative \( a_i \) on the attribute \( j \) is represented by a subset of values \( C^j_{ij} \subseteq X_{ij} \) which depends on the subsets of the nature states (focal elements) \( B^j_h \) (\( h' = 1, ..., 2^H \)) and \( B^j_h \subseteq \Omega^j \).
Table 2
Performance Matrix for the Attribute j Integrating the Prior Information

<table>
<thead>
<tr>
<th>Alternative</th>
<th>Any focal elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_1 )</td>
<td></td>
</tr>
<tr>
<td>( \vdots )</td>
<td></td>
</tr>
<tr>
<td>( \alpha_i )</td>
<td></td>
</tr>
<tr>
<td>( \vdots )</td>
<td></td>
</tr>
<tr>
<td>( \alpha_m )</td>
<td></td>
</tr>
<tr>
<td>Priori belief masses</td>
<td></td>
</tr>
</tbody>
</table>

By using this modeling framework, we could consider that a possibilistic or a probabilistic modeling is a particular case when the prior information is represented by possibility or probability distributions respectively. In fact, when the attribute is a possibilistic one, the prior information related to the evaluation \( e_{ij} \) is characterized by a possibility distributions \( \pi \). In such a context, the corresponding belief masses are associated with focal elements that are embedded \( B_1 \subseteq \ldots \subseteq B_j \subseteq \ldots \subseteq B_H \). The possibility measures coincide with the plausibilities of the embedded focal elements. When the attribute \( j \) is stochastic, the focal elements \( B_h \) are reduced to the singletons \( \{ \omega_h \} \) and the corresponding belief masses correspond to probability measures. In this case, the evaluations are characterized by random variables \( X_{ij}^h \) with prior probability distributions \( f_{ij}^h \). We have then the prior (subjective) probability \( P(\omega_h) \) for each state of the nature \( \omega_h \) (\( h = 1,2,\ldots,H \)).

Local Preference Relations

The first phase of our procedure consists of establishing the local preference relations between two alternatives. Let \( H_j \) be the local relation between two alternatives \( a_i \) and \( a_k \) on a criterion \( j \). We have: \( a_i, H_j a_k \) where \( H_j = \{ P, P^{-1}, Q, Q^{-1}, I, ? \} \) with \( I \) is the indifference relation, \( P \) is the strict preference relation, \( P^{-1} \) is the inverse strict preference relation, \( Q \) is the
weak preference relation, \( Q^I \) is the inverse weak preference relation and \( ? \) is the incomparability relation.

In order to construct these relations, the approach based on the extension of the stochastic dominance concept was privileged. The use of the stochastic dominance concept has the advantage of providing a uniform treatment for the different languages which express the information imperfections. Notice that stochastic dominance allows to conclude about the preference of an alternative \( a_i \) over an alternative \( a_k \) for a decision-maker whose attitude towards risk corresponds to DARA (Decreasing Absolute Risk Aversion) utility functions Martel and Zaras (1995). This type of risk aversion is observed for several economic phenomena. The link between stochastic dominance and the preference is well known for this class of utility functions. It is not always easy to make the decision-maker’s preferences explicit and we can often conclude that \( a_i \) is preferred to \( a_k \) if some stochastic dominance conditions are verified.

In the case where the evaluation \( e_{ij} \) is a random variable, the results of stochastic dominance could be directly applied in order to establish preference relations. For instance, we could say that:

\[
e_{ij} I_{e_{ij}} \iff [H_j(x) = |F_{ij}(x) - F_{kj}(x)| \leq s_j, \forall x \in [x^s_j, x^u_j]] \quad (4)
\]

where \( x^s_j \) and \( x^u_j \) are respectively the inferior and superior limits of the evaluation scale of attribute/criterion \( j \), \( s_j \geq 0 \) is a predetermined threshold, \( F_{ij} \) and \( F_{kj} \) are the cumulated probability distributions for the evaluations of alternatives \( a_i \) and \( a_k \) through an attribute \( j \) and \( x \) is a modality of this attribute.

Otherwise, i.e. for the cases where \( |H_j(x)| > s_j, (F_{ij} \neq F_{kj}) \), we can say that:

\[
e_{ij} F_{ij} \iff e_{ij} \quad FSD_{ij} \quad e_{kj}
\]

\[
e_{ij} Q_{ij} \iff e_{ij} \quad SSD_{ij} \quad or \quad TSD_{ij} \quad e_{kj}
\]

\[
e_{ij} ?_{ij} \iff e_{ij} \quad non\ SD_{ij} \quad e_{kj}
\]

where \( SD \) means that one of the three dominance types is verified. The definitions of the dominance for first order, second order and third order (\( FSD \), \( SSD \) and \( TSD \)) can be found in Martel and Zaras (1995).

These concepts can be extended to ambiguous probabilities (Langewish and Choobineh, 1996). Similarly we can process fuzzy, possibilistic or evidential criteria if we use some known transformations of the functions characterizing the evaluation of an alternative (fuzzy membership functions, possibility distributions, belief masses, …) to give them properties similar to those of a probability density function (Ben Amor et al., 2007).

Let us note that the construction of local preference relations for deterministic criteria can be carried out by using discrimination thresholds to distinguish between strict preference, weak preference and indifference situations (pseudo-criterion notion) in the context of punctual
evaluations. In this case, locally there is no incomparability. It is one of the characteristics of a criterion (Roy, 1985).

Aggregating Local Preference Relations

Once the local preference relations are established, the second stage of our procedure aims at aggregating \( n \) binary relations \((a_j, H a_k)\) in order to obtain a global binary relation \((a_j, H a_k)\) by considering the relative importance of the criteria.

To achieve this, the algorithm (AL3) developed by Jabeur and Martel (2002) in a group decision-making context is adapted to handle the weak preference relation. AL3 starts by defining, for each pair of alternatives \((a_i, a_k)\), an index \(\Phi^H (a_i, a_k)\) where \(H \in \{P, P^{-1}, I, ?\}\). This index measures the divergence between the global relation \(H\) and the local binary relations relating this pair of alternatives on each criterion. This divergence is quantified by using the distance measure \(\Delta\) between two binary relations \(H\) relating two alternatives (Jabeur et al. 2004) when \(H \in \{P, P^{-1}, I, ?\}\).

Including the weak preference and the inverse weak preference relations requires an extended distance measure \(D\) and the following adjustments to the divergence index calculations. The index \(\Phi^H (a_i, a_k)\) is determined as follows:

\[
\Phi^H (a_i, a_k) = \sum_{j=1}^{n} w_j D(H, R_j(a_i, a_k))
\]

where \(R_j(a_i, a_k)\) is the local binary relation comparing \(a_i\) to \(a_k\) on criterion \(j\), \(D\) is the extended distance measure between the binary relations relating the pairs of alternatives and \(H \in \{P, P^{-1}, Q, Q^{-1}, I, ?\}\) is the global relation. In the literature, various methods are proposed to establish the relative importance of the criteria \(w_j\). Regarding this issue, one can refer to Mousseau (1995) and Roy and Mousseau (1996).

Extended Distance Measure \(D\)

Based on the works of Ben Khélifa and Martel (2001) and Jabeur et al. (2004), we propose a set of six “logic” conditions (an axiomatic) allowing to compare the distance between each pair of binary relations \(\{P, P^{-1}, Q, Q^{-1}, I, ?\}\). If \(D\) satisfies these conditions, it can be easily proven that \(D\) is a metric, i.e. it verifies the non-negativity, the symmetry and the triangular inequality axioms.
Condition 1: (C1).
\[
D(P, ?) = D(P^1, ?) \quad \text{and} \quad D(P, I) = D(P^1, I) \quad (7)
\]
\[
D(Q, ?) = D(Q^1, ?) \quad \text{and} \quad D(Q, I) = D(Q^1, I) \quad (8)
\]
This condition is natural since the strict preference \( P \) and the inverse strict preference \( P^1 \), as the weak preference \( Q \) and the inverse weak preference \( Q^1 \) are symmetrical.

Condition 2: (C2).
\[
D(P, I) + D(I, P^{-1}) \geq D(P, P^{-1}) \quad (9)
\]
\[
D(Q, I) + D(I, Q^{-1}) \geq D(Q, Q^{-1}) \quad (10)
\]
\[
D(P,Q) + D(Q, I) \geq D(P, I) \quad (11)
\]
This condition follows from C1 and the triangular inequality requirement for a metric. The first two inequalities indicate that the indifference relation lies between the strict preference and the inverse strict preference relations and between the weak preference and the inverse weak preference relations respectively. The third inequality indicates that the weak preference relation lies between the strict preference and the indifference relations. Only the equality limit cases will be considered here.

Condition 3: (C3).
\[
D(P, P^{-1}) = \max \{D(O, U)/ O, U \in \{P, P^{-1}, Q, Q^{-1}, I, ?\}\} \quad \text{and} \quad (12)
\]
\[
D(Q, Q^{-1}) = \max \{D(O, U)/ O, U \in \{P, P^{-1}, Q, Q^{-1}, I, ?\} \text{ and } (O, U) \neq (P, P^{-1})\} \quad (13)
\]
This condition indicates that the strict preference and the inverse strict preference relations are the most discordant relations and that the weak preference and the inverse weak preference are the second most discordant relations.

Condition 4: (C4).
\[
D(O, U) > 0 \text{ if } O \neq U \quad \text{and} \quad D(O, U) = 0 \text{ if } O \equiv U \quad \text{when} \quad O, U \in \{P, P^{-1}, Q, Q^{-1}, I, ?\} \quad (14)
\]
This condition states that the minimum distance between two distinct relations is positive. It is null in the opposite case.

Condition 5: (C5).
\[
D(P, ?) \geq D(Q, ?) \geq D(I, ?) \quad (15)
\]
This condition states that the passage from indifference to incomparability is less demanding than the passage from preference (or weak preference) to incomparability as stipulated by several authors (Roy and Slowinski, 1993 and Ben Khéïfa and Martel, 2001). Cook et al. (1986), Jabeur et al. (2004) considered that incomparability relation is equidistant from preference and indifference relations.

Condition 6: (C6).
\[
D(I, ?) \geq D(I, P) \geq D(I, Q) \quad (16)
\]
Ben Khélifa and Martel (2001) justified that the distance between indifference and incomparability should be more important than the distance between indifference and preference. The distance between indifference and weak preference is naturally smaller.
Graphically, the previous conditions are illustrated in figure 1. They allow to infer the following partial pre-order:

\[ D(P, P⁻) \geq D(Q, Q⁻) \geq D(P, ?) \geq D(Q, ?) \geq D(I, ?) \geq D(I, P) \geq D(I, Q), \]

and the following relations:

\[ D(P, P⁻)² + D(I, ?)² = D(P, ?)² \] (right triangle property)
\[ D(Q, I)² + D(I, ?)² = D(Q, ?)² \] (17)
\[ D(Q, Q⁻) = 2 D(I, Q) \]
\[ D(P, P⁻) = 2 D(I, P). \]

In order to assign values to these measures, the approach followed by Ben Khélifa and Martel, (2001) is adopted. Thus, the differences between extreme adjacent measures in the previous pre-order are assumed to be equal. That is: \[ D(P, P⁻) - D(Q, Q⁻) = D(Q, Q⁻) - D(P, ?) \]
and \[ D(I, ?) - 1 = 1 - D(I, Q) \]
\[ D(Q, Q⁻) = 2 D(I, Q) \]

Furthermore, \[ D(I, P) = D(I, P) - D(I, Q) \]. Furthermore, \[ D(I, P) \] is arbitrarily set to one, i.e., \[ D(I, P) = 1 \]. It can then be inferred from C1 that \[ D(P, P⁻) = 2 \]. It follows that:

\[ 2 \cdot D(Q, Q⁻) = D(Q, Q⁻) - D(P, ?) \]
\[ D(I, P)² + D(I, ?)² = D(P, ?)² \]
\[ D(I, Q)² + D(I, ?)² = D(Q, ?)² \] (18)
\[ D(I, ?) - 1 = 1 - D(I, Q) \]
\[ D(Q, Q⁻) = 2 D(I, Q) \]

\[ \text{Figure 1} \]

\[ \text{Graphic Representation of Conditions C1 to C6} \]
By solving this linear system we obtain the following values: \( D(I, Q) = 0.88 \), \( D(I, ?) = 1.12 \), \( D(Q, ?) = 1.42 \), \( D(P, ?) = 1.52 \) and \( D(Q, Q^{-1}) = 1.76 \). From C2, one can easily find \( D(P, Q) = 1 - D(I, Q) = 1 - 0.88 = 0.12 \). Table 3 summarizes the values assigned to \( D \).

**Table 3**

**Numerical Values Assigned to the Distance Measure \( D \)**

<table>
<thead>
<tr>
<th>( a_i )</th>
<th>( a_k )</th>
<th>( a_i, I a_k )</th>
<th>( a_i, Q a_k )</th>
<th>( a_i, P a_k )</th>
<th>( a_i, ? a_k )</th>
<th>( a_i, Q^{-1} a_k )</th>
<th>( a_i, P^{-1} a_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_i )</td>
<td>( a_k )</td>
<td>0</td>
<td>0.88</td>
<td>1</td>
<td>1.12</td>
<td>0.88</td>
<td>1</td>
</tr>
<tr>
<td>( a_i )</td>
<td>( Q a_k )</td>
<td>0.88</td>
<td>0</td>
<td>0.12</td>
<td>1.42</td>
<td>1.76</td>
<td>1.88</td>
</tr>
<tr>
<td>( a_i )</td>
<td>( P a_k )</td>
<td>1</td>
<td>0.12</td>
<td>0</td>
<td>1.52</td>
<td>1.88</td>
<td>0</td>
</tr>
<tr>
<td>( a_i )</td>
<td>( ? a_k )</td>
<td>1.12</td>
<td>1.42</td>
<td>1.52</td>
<td>0</td>
<td>1.42</td>
<td>1.52</td>
</tr>
<tr>
<td>( a_i )</td>
<td>( Q^{-1} a_k )</td>
<td>0.88</td>
<td>1.76</td>
<td>1.88</td>
<td>1.42</td>
<td>0</td>
<td>0.12</td>
</tr>
<tr>
<td>( a_i )</td>
<td>( P^{-1} a_k )</td>
<td>1</td>
<td>1.88</td>
<td>0</td>
<td>1.52</td>
<td>0.12</td>
<td>0</td>
</tr>
</tbody>
</table>

Once the divergence indexes have been computed, we identify, for each pair of alternatives \( (a_i, a_k) \), the set of the global relations \( H^\ast \) that minimize the indexes of divergence \( \Phi^H(a_i, a_k) \), that is:

\[
H^* = \left\{ H^* / \Phi^H = \min_{H \in \{P, P^{-1}, Q, Q^{-1}, I, ?\}} \Phi^H(a_i, a_k) \right\}
\]

Therefore, if the set \( H^* \) is reduced to a singleton, i.e. there is only one global relation \( H^* \) that minimizes the indexes of divergence \( \Phi^H(a_i, a_k) \), then \( H^* \) will be retained for the pair of alternatives \( (a_i, a_k) \) globally. In the opposite case, i.e. more that one distinct relations minimize the indexes of divergence \( \Phi^H(a_i, a_k) \), priority rules have to be applied between the binary relations \( \{P, P^{-1}, Q, Q^{-1}, I, ?\} \). We extend the two priority rules proposed by Jabeur (2004) to include weak preference. Jabeur (2004) states that the preference relation priority is higher than that of the indifference relation and that the indifference relation priority is higher than that of the incomparability relation. Likewise, we can consider that the preference relation has higher priority than the weak preference relation, which has higher priority than the indifference relation.

Once the priority rules are applied, we analyze the cardinality of \( H^* \). If \( H^* \) contains a unique relation \( H^* \), then \( H^* \) will be retained for the pair of alternatives \( (a_i, a_k) \) globally. In the opposite case, i.e. when \( H^* \) contains two relations of the same priority (necessarily the preference \( (P) \) and the inverse preference \( (P^{-1}) \) relations or the weak preference \( (Q) \) and the inverse weak
preference ($Q^{-1}$ relations), we extend the intersection rule used by Roy (1978) to combine two complete orders, and therefore, the incomparability relation ($H^* \equiv ?$) will be retained for the pair of alternatives ($a_i, a_k$) globally.

The algorithm AL3 could be summarized as follows:

**Step 1**: Let $B = \{(a_i, a_k) \in A \times A\}$ the set of the alternatives pairs. Compute for all alternatives pairs of $B$ and for each $H \in \{P, P^{-1}, Q, Q^{-1}, I, ?\}$ the divergence index $\Phi^H (a_i, a_k)$.

**Step 2**: Identify for each alternatives pair of $B$ the set of global relations $H^*$. If $H^*$ contains a unique relation, then this relation would be retained for the alternatives pair $(a_i, a_k)$ globally. If not, go to step 3.

**Step 3**: Apply the priority rules. If $H^*$ contains a unique relation, then this relation would be retained for the pair of alternatives $(a_i, a_k)$ globally. If not, go to step 4.

**Step 4**: Apply the intersection rule to retain the incomparability relation for the alternatives pair $(a_i, a_k)$ globally.

Notice that this algorithm requires $n(n-1)/2$ iterations to establish a global preference relational system (p.r.s) from local preference relational systems. Moreover, this algorithm guarantees the independence towards the irrelevant alternatives. Then, adding or removing an irrelevant alternative $a_i$ will not modify in any case the type of the relation between $a_i$ and $a_k$ globally. Once the global p.r.s is determined, it could be exploited according to different decisional problematics (e.g. the choice problematic, the ranking problematic, or the sorting problematic). In the literature, several exploitation procedures can be found. One can quote the works of Roy (1968), Colson (2000) and Jabeur and Martel (2007a) for the choice problematic; the works of Roy (1978), Brans and Vincke (1985), Colson (2000) and Jabeur et al. (2008) for the ranking problematic; and the works of Yu (1992) and Jabeur and Martel (2007b) for the sorting problematic.

**Numerical Example**

Let us consider the following illustrative example for the proposed MCAP: $A = \{a_1, a_2, a_3, a_4\}$ and $X = \{X_1, X_2, X_3, X_4\}$ where $X_1$ is a stochastic attribute, $X_2$ is an evidential attribute, $X_3$ is a possibilistic attribute and $X_4$ is a fuzzy attribute. For each attribute, the inputs consist of the performance matrix, the prior information, and the indifference threshold.

Table 4 gives the performance matrices for the four attributes. The evaluations according to $X_4$ are linguistic variables represented by trapezoidal fuzzy numbers $(a, b, c, d)$ (see figure 2).
Table 4

Performance Matrices for $X_1$, $X_2$, $X_3$, and $X_4$

<table>
<thead>
<tr>
<th>$a_i$</th>
<th>$\Omega^1$</th>
<th>$\Omega^2$</th>
<th>$\Omega^3$</th>
<th>$X_4$</th>
<th>Associated fuzzy numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\omega^1_1$</td>
<td>$\omega^2_1$</td>
<td>$\omega^3_1$</td>
<td>$\omega^4_1$</td>
<td>$\omega^1_2$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>$a_2$</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>$a_3$</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$a_4$</td>
<td>6</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

Figure 2

Membership Function for Fuzzy Trapezoidal Number (a, b, c, d)

Table 5 gives the prior information for attribute $X_1$, while table 6 gives the prior information for $X_2$ and $X_3$. For attribute $X_4$, the prior information is implicitly contained in the performance matrix. We consider the same indifference threshold for all the attributes, i.e. $s_1 = s_2 = s_3 = s_4 = s = 0.05.$
Table 5

A Priori Information for $X_1$

<table>
<thead>
<tr>
<th>$\omega_1^1$</th>
<th>$\omega_2^1$</th>
<th>$\omega_3^1$</th>
<th>$\omega_4^1$</th>
<th>$\omega_5^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(\omega_k^1)$</td>
<td>0.28</td>
<td>0.23</td>
<td>0.10</td>
<td>0.28</td>
</tr>
</tbody>
</table>

Table 6

A Priori Information for $X_2$ and $X_3$

<table>
<thead>
<tr>
<th>$\emptyset$</th>
<th>${\omega_2^2}$</th>
<th>${\omega_2^3}$</th>
<th>${\omega_3^2}$</th>
<th>${\omega_1^2, \omega_2^2}$</th>
<th>${\omega_1^2, \omega_2^3}$</th>
<th>${\omega_2^2, \omega_3^2}$</th>
<th>${\omega_1^2, \omega_2^2, \omega_3^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m(B_k^2)$</td>
<td>0.1</td>
<td>0.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m(B_k^3)$</td>
<td></td>
<td>0.2</td>
<td>0.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Pi(B_k^3)$</td>
<td>0</td>
<td>0.8</td>
<td>0.6</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Local preference relations: Stochastic dominance results could be directly applied in the case of $X_1$. The obtained results and the corresponding local preference relations are given in table 7.

After applying appropriate transformations (Ben Amor et al., 2007), the dominance stochastic results for $X_j$ $j = 2, 3, 4$, led to the local preference relations $H_j$ listed in table 8.

Table 7

Stochastic Dominance Results and Local Preference Relations for $X_1$.

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>*</td>
<td>IND</td>
<td>FSD</td>
<td>?</td>
<td>*</td>
<td>I</td>
<td>P</td>
<td>?</td>
</tr>
<tr>
<td>$a_2$</td>
<td>IND</td>
<td>*</td>
<td>?</td>
<td>-</td>
<td>I</td>
<td>*</td>
<td>?</td>
<td>$Q^{-1}$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>-</td>
<td>?</td>
<td>*</td>
<td>-</td>
<td>$p^{-1}$</td>
<td>?</td>
<td>*</td>
<td>$p^{-1}$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>?</td>
<td>TSD</td>
<td>FSD</td>
<td>*</td>
<td>?</td>
<td>$Q$</td>
<td>$P$</td>
<td>*</td>
</tr>
</tbody>
</table>
Aggregating local preference relations: This step consists in applying the (AL3) algorithm. Thus, we start by computing the divergence index \( \Phi^H(a_i, a_k) \) for each relation \( H \in \{P, P^{-1}, Q, Q^{-1}, I, ?\} \) and for each alternatives pair. This operation is performed by using table 7, table 9 and the distance measure given in table 3. By considering the relative importance vector \( W_j = [4, 2, 3, 1] \), the obtained divergence indexes are given in table 9. For instance, we have:

\[
\Phi'(a_1, a_2) = 0.4D(I, I) + 0.2D(I, P^{-1}) + 0.3D(I, Q) + 0.1D(I, P^{-1}) = (0.4 \times 0) + (0.2 \times 1) + (0.3 \times 0.88) + (0.1 \times 1) = 0.564.
\]

Table 9

Indexes of Divergence

<table>
<thead>
<tr>
<th>Alternative pairs</th>
<th>( \Phi' )</th>
<th>( \Phi^Q )</th>
<th>( \Phi^P )</th>
<th>( \Phi^I )</th>
<th>( \Phi^{Q^{-1}} )</th>
<th>( \Phi^{P^{-1}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (a_1, a_2) )</td>
<td>0.564</td>
<td>0.916</td>
<td>1.036</td>
<td>1.33</td>
<td>0.916</td>
<td>0.964</td>
</tr>
<tr>
<td>( (a_1, a_3) )</td>
<td>0.8</td>
<td>0.448</td>
<td>0.4</td>
<td>1.44</td>
<td>1.504</td>
<td>1.6</td>
</tr>
<tr>
<td>( (a_1, a_4) )</td>
<td>0.4</td>
<td>0.752</td>
<td>0.8</td>
<td>1.28</td>
<td>1.104</td>
<td>1.2</td>
</tr>
<tr>
<td>( (a_2, a_3) )</td>
<td>0.364</td>
<td>0.716</td>
<td>0.836</td>
<td>1.25</td>
<td>1.068</td>
<td>1.164</td>
</tr>
<tr>
<td>( (a_2, a_4) )</td>
<td>0.892</td>
<td>0.716</td>
<td>0.812</td>
<td>1.43</td>
<td>1.068</td>
<td>1.188</td>
</tr>
<tr>
<td>( (a_3, a_4) )</td>
<td>0.964</td>
<td>1.316</td>
<td>1.364</td>
<td>1.49</td>
<td>0.612</td>
<td>0.636</td>
</tr>
</tbody>
</table>

For each alternatives pair, the set of global relations \( H^* \) minimizing the indexes of divergence is identified. In this case, the minimum is unique for each alternatives pair. Therefore, \( H^* \) is reduced to a singleton. The global preference relations are given in table 10. For instance, we have for \( a_1 \) and \( a_2 \):

\[
\text{Min}_{H} \Phi^H(a_1, a_2) = \text{Min} \{0.564, 0.916, 1.036, 1.33, 0.916, 0.964\} = 0.564, \text{thus } H^* = \{I\}.
\]
Let us note that the previous matrix can be exploited according to different decisional problematics as it is proposed by Jabeur (2004).

**Exploitation of the Global Preference Relational System**

The exploitation of the graph associated to the global p.r.s is carried out by using Jabeur and Martel’s (2007a) procedures. These authors propose a detailed review of original procedures to exploit a p.r.s according to three decisional problematics: the choice problematic, the ranking problematic and the sorting problematic. In the case of a choice problematic, for instance, they proposed a procedure based on the alternative performance concept. An extension to this procedure allowing to handle the weak preference relation will be illustrated here.

Any alternative \( a_i \) which verifies the following condition: \( \text{card}(\Psi^-(a_i)) > \text{card}(\Psi^+(a_i)) \) can subsequently be removed from the kernel.

The main idea of this procedure is to build, without erasing any information contained in the graph-prs, a subset of the "best" alternatives, called \( \Theta \), by using their performance (this concept will be defined later). This procedure will be adapted as follows. First, we determine, for each alternative \( a_i \), the following sets:

- **Successors’ set** \( \Psi^{++}(a_i) \) and **predecessors’ set** \( \Psi^{--}(a_i) \):

  \[
  \Psi^{++}(a_i) = \{a_k / a_i Pa_k, i \neq k\} \quad \text{and} \quad \Psi^{--}(a_i) = \{a_k / a_k Pa_i, i \neq k\}, \tag{20}
  \]

- **Weak successors’ set** \( \Psi^+(a_i) \) and **weak predecessors’ set** \( \Psi^-(a_i) \):

  \[
  \Psi^+(a_i) = \{a_k / a_i Pa_k, i \neq k\} \quad \text{and} \quad \Psi^-(a_i) = \{a_k / a_k Pa_i, i \neq k\} \tag{21}
  \]

- the **set of all alternatives with which \( a_i \) is indifferent**:

  \[
  \Psi^*(a_i) = \{a_k / a_i \approx a_k, i \neq k\} \tag{22}
  \]

- The set of alternatives which are incomparable:
\[ N^{isol} = \{ a_i / \Psi^+(a_i) = \Psi^-(a_i) = \Psi^+(a_i) = \Psi^-(a_i) = \emptyset \} \].

Second, we calculate for each alternative \( a_i \in A \setminus N^{isol} \) the index \( \Pi(a_i) \), which measures its performance in the graph-prs. Introducing the weak preference relation, this performance will be defined as follows:

\[
\Pi(a_i) = \text{card}(\Psi^+(a_i)) - \text{card}(\Psi^-(a_i)) + 0.5(\text{card}(\Psi^+(a_i)) - \text{card}(\Psi^-(a_i)))
\]

(24)

Note that the index \( \Pi(a_i) \) considers only the preference relations (strict and weak) for estimating the performance of an alternative \( a_i \). Thus, in order to take into account the effect of the indifference relations in the performance of an alternative \( a_i \), Jabeur and Martel (2007a) add to \( \Pi(a_i) \) the average of the performances of all the alternatives that are indifferent to \( a_i \) in the graph-prs, let:

\[
\hat{\Pi}(a_i) = \begin{cases} 
\Pi(a_i) & \text{if } \text{card}(\Psi^*(a_i)) = 0 \\
\Pi(a_i) + \frac{1}{\text{card}(\Psi^*(a_i))} \sum_{a_k \in \Psi^*(a_i)} \Pi(a_k) & \text{otherwise}
\end{cases}
\]

(25)

It is noteworthy to mention that the additional \( \frac{1}{\text{card}(\Psi^*(a_i))} \sum_{a_k \in \Psi^*(a_i)} \Pi(a_k) \) component allows to correct upward or downward the initial performance of the alternative \( a_i \) expressed by the index \( \Pi(a_i) \). Finally, Jabeur and Martel (2007a) build the \( \Theta \) set in the following manner: \( \Theta = N^{isol} \cup \hat{N} \) where \( \hat{N} = \{ a_i / \text{Max} \hat{\Pi}(a_i) \} \). In other words, \( \Theta \) will contain the alternatives which have obtained the highest performance \( \hat{\Pi}(\ast) \) and those which are incomparable.

The last step of the second exploitation procedure consists in “purifying” \( \Theta \). In this step Jabeur and Martel (2007a) remove from the graph-prs engendered by \( \Theta \), i.e. the subgraph-prs composed of binary relations which link only the alternatives belonging to \( \Theta \), any alternative \( a_k \) such as \( \exists a_k \in \Theta \) and \( a_k \in \Psi^*(a_i) \). If the purified \( \Theta \) is nonempty, then it will contain the “best” alternatives. In the opposite case, i.e. the purified \( \Theta \) is empty, we say that before this operation, \( \Theta \) contained the alternatives which obtained the best score.

**Numerical example (continued)**

On the basis of table 10, we can draw the graph-prs associated to the global preference relations (see Figure 3).

The procedure based on the purified kernel concept starts by the shrinkage operation which consists in identifying circuits included in the graph-prs. There are several circuits and a maximal circuit containing all the alternatives \( \{ a_1, a_2, a_3, a_4 \} \) can be identified. Grouping these alternatives in a single “macro-alternative” \( a' \), leads to a kernel containing one single macro-alternative. It’s important to note that in this numerical example the purification operation is not carried out since the kernel contains only one alternative \( a' \).
In order to identify $\Theta$, i.e. the "best" alternatives set, without erasing the slightest information contained in the graph-prs, the exploitation procedure based on the alternative performance determines, for each alternative $a_i$, the sets $\Psi^+(a_i)$, $\Psi^-(a_i)$, $\Psi^*(a_i)$, $\Psi^x(a_i)$, and the performance indexes $\Pi(a_i)$ and $\hat{\Pi}(a_i)$. All these results are presented in Table 11.

**Figure 3**

Global PRS-graph

![Global PRS-graph](image)

**Table 11**

<table>
<thead>
<tr>
<th>$a_i$</th>
<th>$\Psi^+(a_i)$</th>
<th>$\Psi^-(a_i)$</th>
<th>$\Psi^x(a_i)$</th>
<th>$\Psi^*(a_i)$</th>
<th>$\Pi(a_i)$</th>
<th>$\hat{\Pi}(a_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>${a_3}$</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>${a_2, a_4}$</td>
<td>1</td>
</tr>
<tr>
<td>$a_2$</td>
<td>${a_4}$</td>
<td>---</td>
<td>---</td>
<td>${a_1, a_3}$</td>
<td>0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>$a_3$</td>
<td>---</td>
<td>---</td>
<td>${a_1}$</td>
<td>${a_4}$</td>
<td>${a_2}$</td>
<td>-1.5</td>
</tr>
<tr>
<td>$a_4$</td>
<td>---</td>
<td>${a_3}$</td>
<td>---</td>
<td>${a_2}$</td>
<td>${a_1}$</td>
<td>0</td>
</tr>
</tbody>
</table>

According to table 11, $\Theta$ alternative $a_1$ has obtained the best performance, thus the “best” alternative set will contain $a_1$, i.e. $\Theta=\{a_1\}$. 
Conclusion

The proposed multiple criteria aggregation procedure is designed to consider mixed evaluations. It leads to non-valued global preference relations that could be exploited according to different decisional problematicas. The preference relational system retained contains the strict preference, the inverse strict preference, the weak preference, the inverse weak preference, the indifference and the incomparability relations. The stochastic dominance concepts were adapted accordingly and an extended measure of distance was built. For future work, this distance measure can be improved and the stochastic dominance could also be extended to conclude about the preference for a decision-maker whose attitude towards risk corresponds to INARA (INcreasing Absolute Risk Aversion) utility function (Zaras and Martel, 1994).

References


Roy B., Classement et choix en présence de points de vue multiples (la méthode ELECTRE), *R.I.R.O.* 2 (1968) 57-75.


