CLOSED-FORM TERM STRUCTURE OF GARCH INTEREST RATE

This paper derives a closed form solution for the term structure of two-factor NGARCH short term interest rate by constructing the moment generating function and we also find that Monte Carlo simulation method does not work for the shorter time to maturity because of too much noise.

Introduction

Interest rate is one of the fundamental macroeconomic indicators in both finance and economics models. Capturing the characteristics of the interest rate change and constructing corresponding term structure become more and more important in financial modeling, especially for company valuation and asset pricing. Originating from Merton(1973), who suggested using Brownian diffusion process to model the short term interest rate, many scholars have put lots effort to try to model the term structure of interest in both continues time and discrete time.

From historical data of interest rate, Vasicek (1977) observed that short term interest rate would revert to a certain level in the long run and suggested to use Ornstein-Uhlenbeck process to model the mean reversion property of interest rate. Then Cox, Ingersoll, and Ross (1985) showed that the volatility of interest rate increased with the level of interest rate, which is so-called level effect, and presented CIR model. Incorporating the mean reversion, level effect and stationarity, Chan, Karolyi, Longstaff, and Sanders (1992) constructed a general diffusion process of the change of interest rate. No matter which parameters they put in the drift component or random component, these models are all one-factor model in which the instantaneous interest rate is the only stochastic factor. Does one-factor enough to capture all the characteristics of the interest rate change and provide us with a term structure closed enough to the real term structure which we observed? Although some of one factor models could present a very elegant closed-form solution for the term structure in continues time and also easily to estimate when applied in the empirical work, there still are several serious drawbacks we can’t ignore. As the instantaneous interest rate is the only factor in these models, this implies that the instantaneous returns on bonds of all maturities are perfectly correlated\(^1\) and that there is only parallel shift of the yield curve. In addition, adding the level effect in the diffusion component will lead to overemphasize the sensitivity of volatility to change of interest rate. What’s more important, many empirical works show that the term structure provided by one-factor model is far away from real historical term structure.

---

\(^1\) Longstaff and Schwartz(1992)
In order to solve the problems raised by one-factor model, the multifactor models came through. Cox, Ingersoll, and Ross suggested that we can add some independent factors in one factor model theoretically. Afterwards, Longstaff and Schwartz (1992) used a two factor model to model the diffusion process of short term interest rate, and the two factors they used are instantaneous interest rate and volatility of instantaneous interest rate. Heston, Nandi (1998) presented a discrete time model with GARCH volatility to represent the diffusion process and then derived a closed form solution for pricing bonds and bond options. They found that two-factor model does not improve the pricing of cross section of spot bonds, statistically and economically. In their work, they provided a method to get the close-form solution with GARCH interest rate. However, their focus was on option pricing and did not provide a deep analysis of the term structure of interest rate. In this paper, we will use Heston-Nandi’s GARCH framework and derive a closed form solution for the term structure of interest rate. The structure of this paper is organized as follows. In section 2, we will introduce our GARCH interest rate model and derive a closed form solution using Heston and Nandi’s methodology. We also show how to get the term structure of the two-factor interest rate using Monte Carlo simulation method. In section 3, we will use weekly 3 month US treasury bills from Jan, 1971 to Jan 2008 to do the empirical test and compare the results of closed-form solution and Monte Carlo simulation. We will show how the parameters in the drift and diffusion process affect the shape of interest rate term structure in this section as well. In section 4, we will make conclusion.

GARCH Interest Rate Model and Term Structure

GARCH Interest Rate Model

Following Heston and Nandi’s two factor interest rate model structure, we construct our two factor GARCH interest rate model in the following form:

\[ r_{t+1} = r_t + \kappa (\theta - r_t) + \sqrt{h_{t+1}} \times w_{t+1} \]  \hspace{1cm} (1)

\[ h_{t+1} = \varphi + \beta h_t + \alpha (w_t - \gamma \sqrt{h_t})^2 \] \hspace{1cm} (2)

Where, \( r_t \) is the instantaneous interest rate at time \( t \). \( \theta \), is the long run interest rate. \( \kappa \), is the speed of mean reversion to the long run interest rate. \( h_{t+1} \), is the variance of instantaneous interest rate at time \( t+1 \). \( \varphi, \beta, \alpha, \gamma \) are the parameter in the stochastic process of conditional variance of instantaneous interest rate. \( W \) is the innovation drawing from standard normal distribution.

In equation (1), the change of interest rate is controlled by two factors, which are instantaneous interest rate and volatility of instantaneous interest rate. We keep the mean reversion parameter in the drift, which means the change of interest rate will revert back to a certain level in the long run at speed \( \kappa \). Equation (2) shows the stochastic diffusion process of conditional variance of interest rate, which follows asymmetric NGARCH (1,1) process. Although equation (2) is not exactly the general NGARCH expression\(^2\), it has all the properties of NGARCH model. For example, \( \gamma \) controls the asymmetric effect of this GARCH process. According to the empirical work, we always find that higher interest rate level accompanies with higher volatility and lower interest rate level accompanies with lower volatility, which

\(^2\) Usually, NGARCH is written in this way: \( h_{t+1} = \omega + \beta h_t + \alpha (z_t + \theta)^2 \)
mean positive relationship between interest rate level and volatility level. Therefore, we expect to see the value of $\gamma$ be negative. Our expression is the same as Heston and Nandi (1998)'s GARCH model. In their paper, Heston and Nandi derived the closed form solution for the interest rate option using this expression. They also proved that this GARCH process will converge to the continue time if the time intervals are small enough. So, we prefer to call is HNGARCH. We use this expression so as to apply Heston and Nandi's trick to derive the closed form solution for the term structure of short term interest rate in discrete time. While for the unconditional volatility, we can take the expectation of equation (2) on both sides and get following results:

$$E(h) = \frac{\varphi + \gamma^2}{1 - \beta - \alpha \gamma^2} \quad (3)$$

As the variance cannot be negative, we can get the persistent level $\beta + \alpha \gamma^2 < 1$ and also $\varphi + \alpha > 0$. We need to add these constraints when we do the parameter estimations in section 3.

Closed-Form Solution

Using Heston and Nandi (1998) method, we can derive a closed form solution for the term structure of interest rate. Before we do the derivation, we need to show the relationship between price of zero coupon bonds and the instantaneous interest rate, which is equation (4) and (5).

$$D(t, T) = e^{-\hat{\rho}(T-t)} = e^{-\Sigma_\tau r_{\tau}\Delta t} \quad (4)$$

$$D(t, \tau)e^{-\tau \Delta t} = D(t, \tau + \Delta), \text{ where } \tau < T \quad (5)$$

Where $D(t, \tau)$ is the price of 1 dollar zero coupon bonds at time $t$ which mature at time $\tau$ with 1 dollar face value, $\hat{\rho}$ is the equivalent yield to maturity. From equation (5), we take log to both sides of the equation and get equation (6).

$$lnD(t, \tau) - \tau \Delta t = lnD(t, \tau + \Delta) \quad (6)$$

Then we construct the moment generating function:

$$f(t, T, \phi) = E_\tau[\exp(\phi \ln(D(t, T)))] \quad (7)$$

We guess the solution for this moment generating function is log-linear and the form is following:

$$f(t, T, \phi) = \exp(\phi \ln(D(t, T))) + A(t, T, \phi) r_{\tau} + B(t, T, \phi) h_{\tau} + C(t, T, \phi)$$

The terminal condition is following:

$$D(t, t) = 1, \quad A(t, T, \phi) = B(t, T, \phi) = C(t, T, \phi) = 0$$

We can solve the undetermined coefficients by iterated expectations:

$$f(t, \tau, \phi) = E_\tau * [f(t, \tau + \Delta, \phi)]$$

$$= E_\tau[\exp(\phi \lnD(t, \tau + \Delta) + A(t, \tau + \Delta, \phi)r_{\tau+1} + B(t, \tau + \Delta, \phi)h_{\tau+1} + C(t, \tau + \Delta, \phi))] \quad (8)$$
Inserting equation (1), (2) and (6) into equation (8), substituting the dynamics and rearranging term, we can get:

\[
f(t, \tau, \phi) = E_t \left[ \exp \left( \phi \ln D(t, \tau) - \phi r_\tau + A(t, \tau + \Delta, \phi) r_\tau - A(t, \tau + \Delta, \phi) \kappa r_\tau + A(t, \tau + \Delta, \phi) \kappa \theta 
+ B(t, \tau + \Delta, \phi) \varphi + B(t, \tau + \Delta, \phi) \beta h_\tau + B(t, \tau + \Delta, \phi) \alpha \gamma^2 h_\tau 
+ \frac{(A(t, \tau + \Delta, \phi) - 2B(t, \tau + \Delta, \phi) \alpha \gamma)^2 h_\tau}{4B(t, \tau + \Delta, \phi) \alpha} \right) \right]
\]

As \( W_t \) is the innovation with standard normal distribution, we have following relationship.

\[
E[\exp(\alpha(W_t + b)^2)] = \exp(-0.5 \ln(1 - 2\alpha) + \frac{ab^2}{1 - 2\alpha})
\]

Using above relationship, we can solve the moment generating function and get:

\[
f(t, \tau, \phi) = \exp[\phi \ln D(t, \tau) - \phi r_\tau + A(t, \tau + \Delta, \phi) r_\tau - A(t, \tau + \Delta, \phi) \kappa r_\tau + A(t, \tau + \Delta, \phi) \kappa \theta 
+ B(t, \tau + \Delta, \phi) \varphi + B(t, \tau + \Delta, \phi) \beta h_\tau + B(t, \tau + \Delta, \phi) \alpha \gamma^2 h_\tau 
- \frac{1}{2} \ln(1 - 2B(t, \tau + \Delta, \phi) \alpha) + \frac{(A(t, \tau + \Delta, \phi) - 2B(t, \tau + \Delta, \phi) \alpha \gamma)^2 h_\tau}{2(1 - 2B(t, \tau + \Delta, \phi) \alpha)} 
+ C(t, \tau + \Delta, \phi)]
\]

Therefore, the coefficients of the moment generating function are:

\[
A(t, \tau, \phi) = A(t, \tau + \Delta, \phi) - \phi - A(t, \tau + \Delta, \phi) \kappa
\]

\[
B(t, \tau, \phi) = B(t, \tau + \Delta, \phi) \beta + B(t, \tau + \Delta, \phi) \alpha \gamma^2 + \frac{(A(t, \tau + \Delta, \phi) - 2B(t, \tau + \Delta, \phi))^2}{2(1 - 2B(t, \tau + \Delta, \phi) \alpha)}
\]

\[
C(t, \tau, \phi) = A(t, \tau, \phi) \kappa \theta + B(t, \tau + \Delta, \phi) \varphi - \frac{1}{2} \ln[1 - 2B(t, \tau + \Delta, \phi) \alpha] + C(t, \tau + \Delta, \phi)
\]

Now, we can calculate coefficients back to \( A(t, t, \phi), B(t, t, \phi), C(t, t, \phi) \) using terminal condition and \( f(1) \) is the expected value of \( D(t, T) \). Given the price of zero coupon bonds with 1 dollar face value, we can compute the equivalent yield to maturity straightforward.

**Monte Carlo Simulation**

Besides closed form solution, we can also use Monte Carlo simulation method to simulate the path of interest rate and compute the expected value of the yield to maturity. Then we can get the term structure of GARCH interest rate as well. In this section, we will show how to use Monte Carlo method to construct the term structure with GARCH interest rate diffusion process.
At time 0 (current time), we can observe the instantaneous interest rate $r_0$ and get the instantaneous variance of interest rate, $h_0$, using historical data.

Step one: draw innovation $W_0$ from standard normal distribution and compute $h_1$ from equation (2), given $h_0$.

Step two: draw another innovation $W_1$ from standard normal distribution. Given $h_1, r_0$, we can compute $r_1$ from equation (1).

Step three: repeat step one and step two until we reach maturity time $T$. Now we have all discrete time $r_t$, and compute $D(t, T) = e^{-\sum_t r_t \Delta t}$.

Step four: repeat step one, two and three 1000 time. Then we can calculate the average value of $D(t, T)$, which is the expected price of zero coupon bond with 1 dollar face value.

Empirical Results

Data

In this paper, we use weekly observations for the annualized yield on three month U.S. treasury bills. All the data is downloaded from H.15 database\(^3\). The time period of our data is from January, 1971 to January, 2008. The summary statistic of our dataset is showed in table 1. We can see that the mean of annualized interest rate during 1971 to 2008 is around 6%. The lowest interest rate is 0.8% and the highest interest rate is 16%, which is a huge difference.

Obviously, the interest rate is not constant, but fluctuates dramatically. We also found that the autocorrelation of interest rate is pretty high, however, when comes to the change of interest rate, the autocorrelation is only 0.27. So assuming that the innovations in the diffusion process are independent with each other is not a very strong assumption. In addition, we can see that the skewness of change of interest rate is not a big difference from zero, and the excess kurtosis is 3.32. The true distribution of the innovations should be a little different form normal distribution. However, in order to get a closed form solution, we assume the innovations of interest rate change follow a normal distribution.

Table 1

<table>
<thead>
<tr>
<th>Summary Statistic of dataset</th>
<th>interest rate level</th>
<th>Interest rate change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observations</td>
<td>1930</td>
<td>1929</td>
</tr>
<tr>
<td>mean</td>
<td>0.059116528</td>
<td>0.000226004</td>
</tr>
<tr>
<td>standard deviation</td>
<td>0.028953819</td>
<td>0.029600272</td>
</tr>
<tr>
<td>variance</td>
<td>0.000838324</td>
<td>0.000876176</td>
</tr>
</tbody>
</table>

\(^3\) The website of H15 database is: http://www.federalreserve.gov/releases/h15/update/
MLE Estimation and empirical results

In order to get the term structure from the GARCH diffusion process, we need to estimate the parameters based on the historical data. In this study, we use annualized weekly 3-month treasury-bill. As the time interval for our discrete time model is one week, we have to convert annualized 3-month treasury-bill rate into weekly interest rate by dividing 524. Then we can write the likelihood function assuming normality:

\[ R_{t+1} = r_{t+1} - r_t - \kappa(\theta - r_t) = \sqrt{h_{t+1} \cdot W_{t+1}} \]

\[ h_{t+1} = \varphi + \beta h_t + \alpha(W_t - \gamma\sqrt{h_t})^2 \]

\[ \text{Likelihood Function} = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi h_t}} \exp\left(-\frac{R_t^2}{2h_t}\right) \]

We use Matlab to do the maximum likelihood estimation. After trying different starting value, we got the value of parameters in Table 2. We can see that all the parameters are significant from zero with 1% significant level. The long run interest rate, \( \theta = 0.00113 \), approximately equal to the mean of our data sample, which is 0.001137. This tells us that the long run interest rate level will revert back to 0.113% (annualized is 5.91%)5. The speed of reversion, \( \kappa = 0.004025 \), is significant, showing us that the mean reversion effect does exist in the historical interest rate diffusion process. While for the

| Parameter Estimation Using MLE |
|-------------------------------|-----------------|--------------|----------|--------|------|----------------|
| \( \kappa \)                  | \( \theta \)    | \( \varphi \) | \( \beta \) | \( \alpha \) | \( \gamma \) | Likelihood    |
| Parameters                    | 0.004025        | 0.00113      | 1E-11    | 0.72   | 2.8E-10 | -258.25       | 17521.67      |
| t-statistics value            | -6.5            | -6.37E9      | -1.2E4   | -1.47E3| -1.07E6 | 5.9           |               |

4 We assume that there is 52 weeks in one year.
5 We calculate the annualized interest rate by multiplying 52.
parameters in the volatility diffusion process, we found that the asymmetric parameter, \( \gamma = -258.25 \), which exhibited positive relationship between the interest rate level and volatility. This means higher interest rate level should accompany with higher volatility level and vice versa. Then we take a look at the persistent level of the volatility of interest rate, which is \( \beta + \gamma = 0.72 < 1 \). Therefore the unconditional variance of interest rate is:

\[
E(h) = \frac{\varphi + \alpha}{1 - \beta - \gamma^2} = 1.046E - 9
\]

Although compared to the persistent level of stock return, which is around 0.9, the persistent level of short term interest rate is a little lower, we have to say that the volatility interest rate is still very persistent. We can see the volatility clustering phenomenon of short term interest rate from Figure 1.

Putting the value of parameters into our closed form solution, we can draw the term structure of interest rate, which is showed in Figure 2. As there are two factors in our diffusion process, one is instantaneous interest rate and the other is the volatility of instantaneous interest rate. In Figure 2, we use the sample mean, 0.06, and sample variance 1.1E-7 as the instantaneous interest rate and instantaneous variance of interest rate, respectively. We can see that the term structure of interest rate is down sloping. As the horizontal axis is the number of weeks, we need to divide the number of weeks by 52 to get the number of years. For example, the annualized yield to 3 year maturity is around 150 weeks and is approximately 5.9%.
We draw these three pictures using the weekly, three month U.S. short term interest rate data from January, 1971 to January, 2008. The top graph shows the time-varying interest rate. The middle graph shows the time-varying change of interest rate and the bottom graph show the time-varying volatility of interest rate.
Figure 2

Price of Zero Coupon Bond with Corresponding Term Structure of Interest Rate

Price of zero coupon bond

Term Structure of interest rate

Yield to Maturity
Figure 3
Fixed Variance of Interest Rate at 1.1E-7 and Change Instantaneous Interest Rate

Figure 4
Fixed Instantaneous Interest Rate at 6% and Change Variance of Interest Rate
In order to check the relationship between instantaneous interest rate (or volatility of interest rate) and the shape of term structure. We fixed the instantaneous variance of interest rate and change instantaneous interest rate, which is showed in Figure 3. From Figure 3, we found that the term structure is up sloping when the instantaneous interest rate is lower than the long run interest rate, and the term structure is down sloping when the instantaneous interest rate is higher than the long run interest rate, which is exactly so-called mean reversion effect. Many scholars have found the mean reversion effect of interest rate already, such as Vasicek (1977), Cox, Ingersoll, and Ross (1985), etc. Besides this, another interesting finding is presented in Figure 4 and 5. In Figure 4, we fix the instantaneous interest rate at 6% but change the instantaneous variance of interest rate. What exactly we do is that we times the sample variance by 10 and divided the sample variance by 10 as well. Now we can see three lines in Figure 4 with different instantaneous variance of interest rate. We found that when instantaneous variance of interest rate is higher, the current price of zero coupon bond is higher with the same maturity, and the shape of term structure is more steeper. This means higher instantaneous variance of interest rate will make the current price of zero coupon bond higher. If we fix the instantaneous interest rate at 3%, which is lower than long run interest rate, we found the price of zero coupon bond is the highest with the highest instantaneous variance of interest rate as well. But for the shape of term structure, higher instantaneous variance will make the term structure of interest rate more flat, which is showed in Figure 5. Another thing we need to notice is that when we enlarge the variance 10 times from 1.1E-8 to 1.1E-7, the term structure of interest rate becomes steeper, but the shape does not change a lot. However, when we enlarge the variance 10 times form 1.1E-7 to 1.1E-6, the shape of term structure changes quite a lot. This maybe due to the absolute change of variance is different even though we scale the variance up or down with the same size.
Closed-Form Solution with Monte Carlo Simulation

Figure 6
Closed Form Solution VS Monte Carlo Simulation

In this section, we compare the results from closed form solution with the results from Monte Carlo simulation, which is showed in Figure 6. In Figure 6, we set the instantaneous interest rate at 6% and get the price of zero coupon bonds and the corresponding term structure. On the top graph, we can see that the line from closed form solution and the line from Monte Carlo simulation nearly overlap, which means that the answer from our closed form solution should be reliable. While for the bottom graph, if the time to maturity is low, we cannot get a stable answer from Monte Carlo simulation because fewer number of draw from standard normal distribution will generate too much noise. The noise becomes much lower with longer the time to maturity. In Figure 6, we can see clearly that after 50 weeks the two methods generate very similar number. Therefore, compare to Monte Carlo simulation method, our closed form solution not only can save lots of computation time, but also can generate reliable results for short time to maturity.

Conclusion

According to Bali (2000)’s paper, we know that two factor model of interest rate fit the historical interest rate much better than the one factor model, especially for the NGARCH interest rate model. In this paper, we started from Heston and Nandi’s expression for NGARCH model. Then using moment generating function and Heston and Nandi's trick, we derived a closed form solution for the term structure of short term interest rate. After that, we also showed how to simulate the diffusion process using Monte Carlo method.
Empirically, we use US weekly 3-month treasury bills rate from January, 1971 to January, 2008 to do the empirical test on our solution. There are three main findings. Firstly, we confirmed the mean reversion property of short term interest rate by fixing the volatility and changing instantaneous interest rate. In our example, if we fixed the long run interest rate at 5%, we will get a downward sloping interest rate term structure when we set the instantaneous interest rate larger than 5%, and vice versa. Secondly, we found that the zero coupon bond should be more valuable with higher instantaneous volatility of interest rate if we fixed the instantaneous interest rate and time to maturity. The last finding was that Monte Carlo simulation can generate similar term structure of interest rate with longer time to maturity; however, Monte Carlo simulation method does not work for the shorter time to maturity because of too much noise.

References


